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The categorical product of two odd cycles is a 2-connected graph

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ABSTRACT

The concept of the categorical product of graphs was introduced by the mathematician Frank Harary in his paper titled "Topological Concepts for Maximum Graphs" published in 1969. The categorical product of graphs can be used to model and analyze complex network topologies. By considering the product of individual components, such as nodes or edges, one can study the interconnections and properties of the overall network structure. In light of this, our study aims to explore the properties of the categorical product of two graphs and demonstrate that the categorical products of two cycle graphs with odd lengths are 2-connected.

Keywords *– graph theory, graph, product of graph, categorical product of graphs.*

1.Preliminaries:

This section aims to present the necessary definitions to clarify our main result, starting with the foundational definition of a graph.

Definition: A graph G = (V, E) is a mathematical structure defined as a set V of vertices and a set E of edges. The edges in E are composed of pairs of vertices from V. To denote the vertex set and edge set of graph G, V(G) and E(G) are commonly used, respectively.

Definition： Given graph G=(V,E) and H=(W,K), their categorical product is defined as G×H = (V×W,M), where an edge {(u,v),(w,k)} is in M=E(G×H) if and only if {u,w} is an edge in E(G) and {v,k} is an edge in E(H).

Example: The categorical product of the following two path graphs consists of two connected components:

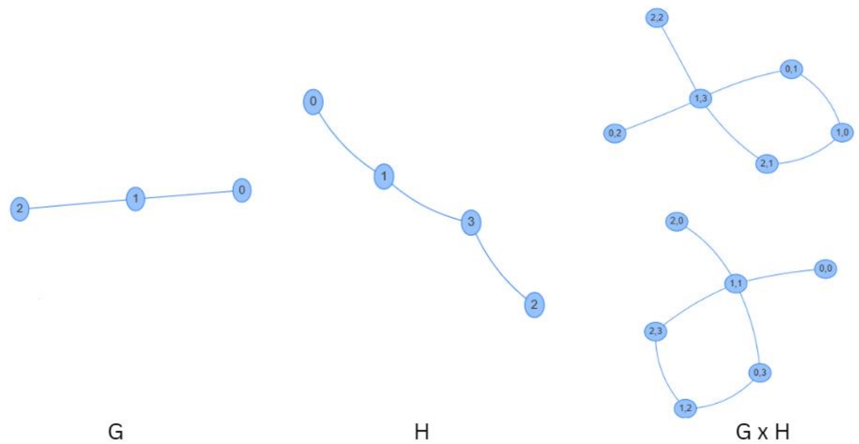

Figure 1

Definition: If a graph G requires the removal of at least k-vertices to become disconnected, it is referred to as k-connected.
Example: A cycle graph with a length greater than 3 is considered 2-connected since removing any single vertex cannot disconnect the graph, and removing two non-adjacent vertices will lead to graph disconnection.

2. MAIN RESULT

Lemma1: The number of edges in the Cartesian product of graphs G and H, is equal to twice the product of the number of edges in G and the number of edges in H.

Proof: Each pair of edges {a,b} in G and {c,d} in H corresponds to two edges in the Cartesian product G× H: {(a,c),(b,d)} and {(a,d),(b,c)}. The number of edges in G×H can be determined by counting all the possible pairs of edges in G and H, which is equal to 2 times the product of the number of edges in G and the number of edges in H. Hence, the total number of edges in G times H is 2*|E(G)|*|E(H)|.

Lemma 2: The categorical product of the two trees T_1 and T_2 comprises of two connected components, rendering it disconnected.

Proof: We first prove that the categorical product of two trees, T1 and T2, is disconnected with two connected components. This can be observed by considering a vertex u in T1 and an edge (v,w) in T2. In this case, there exists no path from (u,v) to (u,w) in the categorical product T1 times T2. The reason for this is that all paths from u to itself in T1 have an even length, while all paths from v to w in T2 have an odd length. Moreover, the presence of a path from (a,b) to (c,d) in T1 times T2 is dependent on the existence of a walk W1 from a to c with the same length as a walk W2 from b to d in T2. Given that it is not possible to construct both W1 and W2 with the same length, it follows that there is no path from (u,v) to (u,w). Moreover, there are precisely two connected components in the graph T1 times T2. This is due to the fact that for any vertex (a, b) in T1 times T2, we can create a walk of equal length from u to a and from v to b, or we can create a walk of equal length from u to a and from w to b. As a result, we can infer that there exists either a path from (u, v) to (a, b) or a path from (u, w) to (a, b). Consequently, T1 times T2 consists of two connected components.

We can see from the example in figure2 that the categorical product of two tree is indeed two disconnected graph.

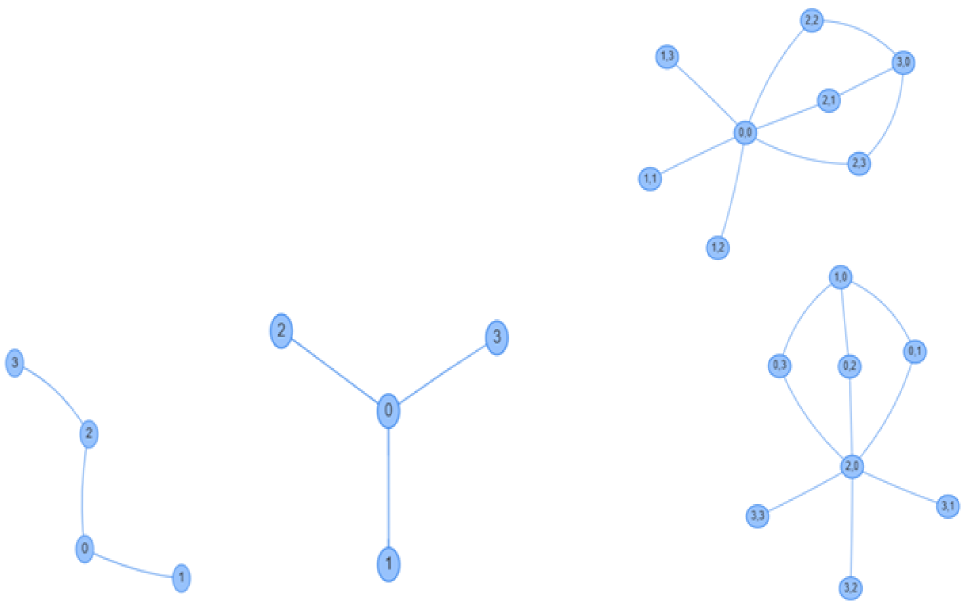

Figure 2

Lemma3: The categorical product of two cycles C1 and C2 of even length is disjoined into two connected components.

Proof: If u is a vertex in C_1 and (v,w) is an edge in C2, then employing a similar reasoning as in lemma1 reveals the impossibility of establishing a path connection from (u,v) in V(C1×C2) to (u,w) in $V(C_1 \times C_2)$, in C1 × C2. Consequently, the graph C1×C2 becomes disconnected. Furthermore, based on a similar argument as in lemma 1, it can be deduced that C1×C2 consists of two connected components.

Example: Please refer to Figure 3 for the categorical product of C4 and C6. It is worth noting that the result is disconnected, consisting of two distinct connected components.

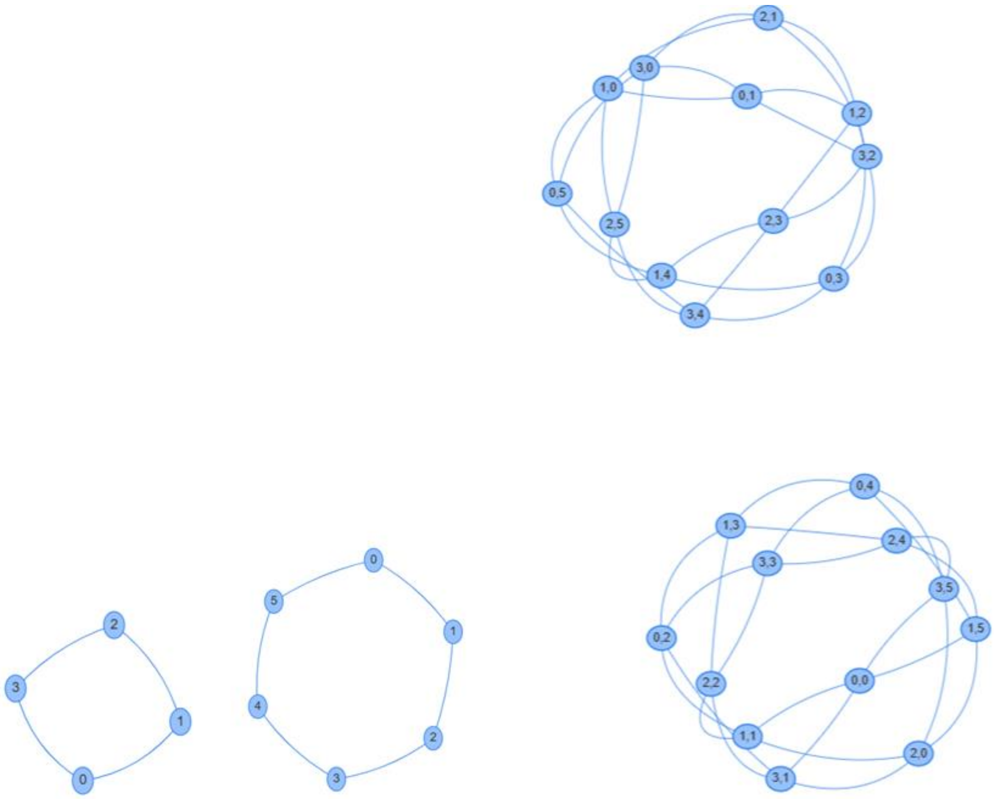

Figure 3

Lemma4: The categorical product of an odd cycle C1 and a connected graph G is connected.
Proof: Consider two vertices (u, v) and (w, x) in the categorical product of graphs V(C1 × G). Let P be a path from v to x in G. Without loss of generality, let's assume that the length of P is odd. We can find a path P' from u to w in C1 of odd length. By walking back and forth between P and P' if necessary, we can construct a walk w1 from u to w in C1 and a walk from v to x in G of the same length. These walks can then be combined to form a path from (u, v) to (w, x) in the categorical product C1 × G.

Example: Please refer to Figure 4 for the categorical product of C3 with the Turán graph T(6,3). It is worth noting that the resulting graph is connected.

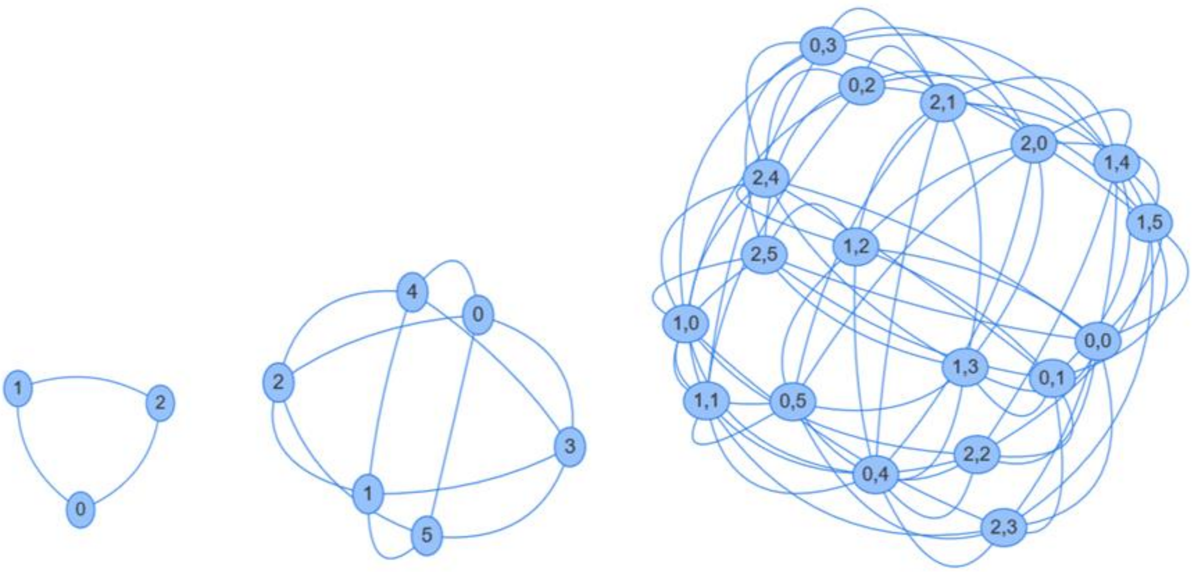

Figure 4

Lemma5: Suppose G is an n-regular graph and H is an m-regular graph. Then, the graph formed by the Cartesian product of G and H,G×H, is an n*m-regular graph.

Proof: If the vertex (a,b) belongs to G×H, then there exists an edge from (a,b) to (c,d) for every ordered pair (c,d), where c is an element of N(a) and d is an element of N(b). Thus, the number of incident edges of (a,b) is equal to the product of the degrees of a and b, denoted as deg(a) and deg(b) respectively, which is also equal to n*m.

Theorem1 : Let C1 and C2 by two cyclic graph of order n and m respectively. If n is odd and m is odd then the categorical product of C1 and C2 is a 2-connected graph.

Proof: To demonstrate that removing any vertex from C1 times C2 still keeps the graph connected, let's assume we remove the vertex (e,b). We want to prove, without loss of generality, that the resulting graph still has a path from (e,a) to (d,c) that doesn't go through (e,b).
Let P be a path from (e,a) to (d,c) in C1 times C2, and let w1 and w2 be the correponding walks from e to d in C1 and from a to c in C2, respectively.
See following figure for an example:

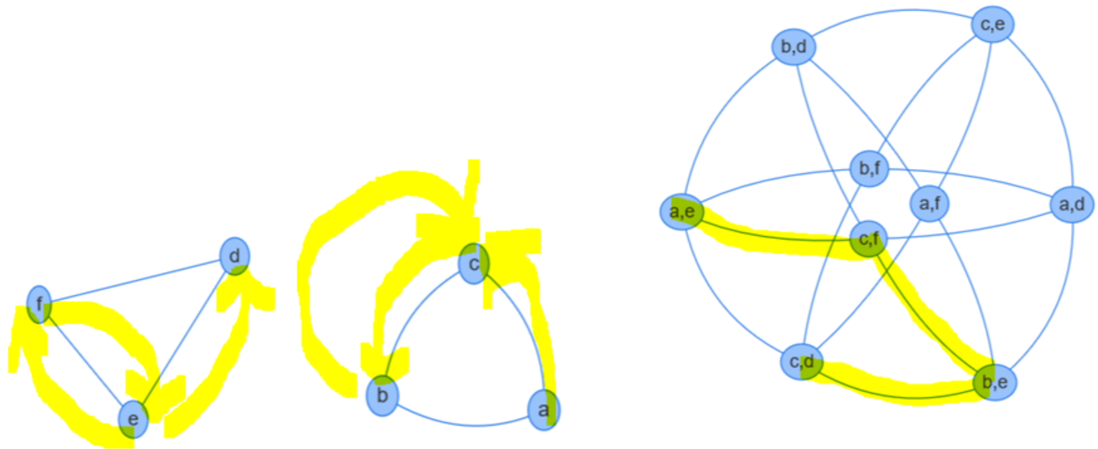

Figure 5

We will construct a new path, P2, from (e,a) to (d,c) that does not use (e,b).
Consider w1: e=a(1),a(1),…,a(i),e,a(i+1),…,d and w2: a=b(1),b(2),…,b(i),b,b(i+1)… ,c.
We will modify w1 to be …,a(i),a(i-1),a(i),e,a(i+2)... and w2 to be …,b(i),b,b(i+1),b(i+2),b(i+1)… .
If a(i-1) does not exist in w1 (assuming i = 1), we can select the vertex adjacent to a(i) in the opposite direction of a(i+1). This is feasible because C1 is a cycle graph. Please see Figure6 for an example of changing the paths:

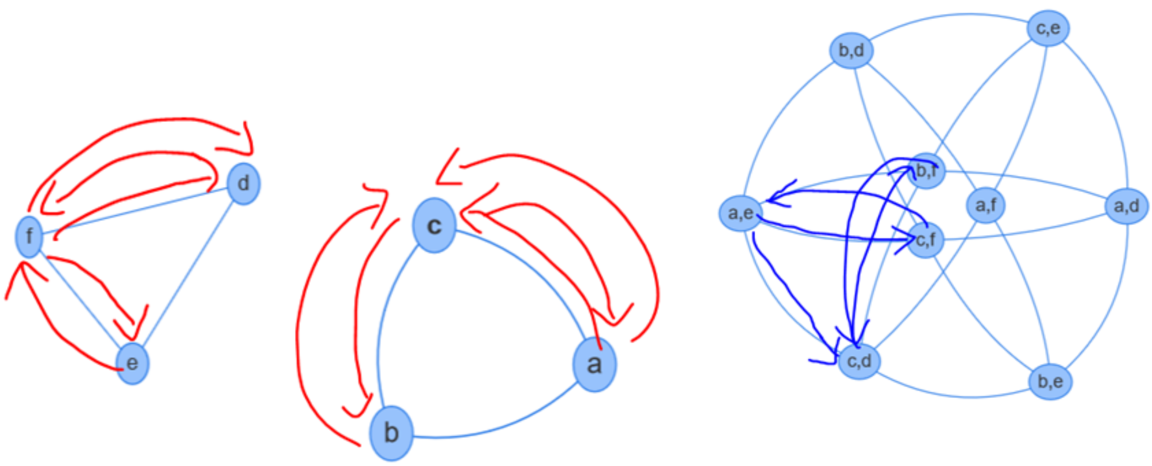

Figure 6

By making these modifications, the resulting walks will give a path P2 in C1×C2 from (e,a) to (d,c) that does not pass through (e,b). It is important to note that the condition that n and m are odd is crucial in this context. If this condition is not met, the graph would be disjointed and consist of two separate connected components, as illustrated earlier.

3.CONCLUSION

The concept of graph categorical product has been introduced and its connectivity properties have been explored. Moving forward, our intention is to generalize the properties that have been identified.

4. REFERENCES

[1] Frank Harary, The maximum Connectivity of a graph. (National Academy of Sciences of the United States of America Jul. 15 1962), Vol.48, No.7 pp.1142-1146.

