ABSTRACT

The concept of open pyramid graph has been introduced and it has been established that it is closely related to BCK-algebra and Euler circuit.

Keywords— Euler Circuit, Open Pyramid Graph, BCK Algebra, and Disjoint Elements

1. INTRODUCTION

Definition (1.1) : A BCK – algebra is a system (E, *, 0) having a non empty set E, a binary operation * and a fixed element 0 such that elements x, y, z \in E satisfy the conditions:

(i) 0 * x = 0
(ii) x * 0 = x
(iii) ((x * y) * (x * z) * (z * y) = 0
(iv) x * y = 0 = y * x \!’ x = y.

Definition (1.2) : A pair \{x, y\} of distinct elements of E is said to be

(a). mutually disjoint if x * y = x and y * x = y.
(b) Semi mutually disjoin if either x*y = x , y*x=0 ,
or y*x = y, x*y=0

Rashmi Rani and Puja (2021) has established the following useful result

Theorem (1.3) : Let E = \{ 0 \equiv \ u_0 , u_1 ,..., u_{n-1} \} and let * be a binary operation defined on E such that

0 * u_i = 0, u_i * 0 = u_i, u_i * u_i = 0

For i=1,2,3,...........,n-1

We also define \ u_i * u_j = u_i and u_j * u_i = 0 for i < j

or u_i * u_j = 0 and u_j * u_i = u_j i , j = 1, 2, ......, n-1. Then

(E, *, 0) is a BCK algebra .

Corollary ( 1.4 ) Under the conditions of theorem (1.3) if we take some pairs as mutually disjoint then the result also hold.

Definition(1.5):- Let G = (V, E) be a graph. A path in G is called an Euler path if it includes every edge exactly once. Further, if it is also a circuit then it is called an Euler circuit.

Example(1.6):- We consider the graphs

Graph (1.1)  

Graph (1.2)
For the graph (1.1) Euler circuit is
\[ a_1 (a_1 a_2) a_2 (a_2 a_3) a_3 (a_3 a_4) a_4 (a_4 a_5) a_5 (a_5 a_6) a_6 (a_6 a_1) a_1 \] (1.2)

For the graph (1.2), we have Euler circuits as
\[ A (AB) B (BO) O (OC) C (CD) D (DO) O (OA) A \] (1.3)
and \[ A (AO) O (OD) D (DC) C (CO) O (OB) B (BA) A \] (1.4)

So Euler circuit of a graph is not unique.

**Definition (1.7)** - Let \( G = (V, E) \) be a simple graph where \( V = \{ a_0, a_1, a_2, \ldots, a_n \} \), \( n = 2m \). If there exists an element \( a_k \) such that \( d(a_k) = n \) and \( d(a_k) = 2 \) for \( i = 1, 2, \ldots, n \) then \( G \) is called an open pyramid graph of order \( (n + 1) \) with vertex \( a_0 \) and base points \( a_0, a_1, a_2, \ldots, a_n \). In this case edges of an open pyramid graph are \( \{ a_0, a_1 \}, \{ a_0, a_2 \}, \ldots, \{ a_0, a_n \}, \{ a_1, a_2 \}, \ldots, \{ a_1, a_n \}, \{ a_2, a_3 \}, \ldots, \{ a_n, a_0 \} \).

Here \( n \) must satisfy \( n \geq 4 \).

**Example (1.8)** - For \( n = 6 \), the open pyramid graph is as follows:

Here \( V = \{ a_0, a_1, a_2, a_3, a_4, a_5, a_6 \} \) where \( n = 6 \) and \( m = 3 \)

This open pyramid graph has Euler circuit as
\[ a_1 (a_1 a_2) a_2 (a_2 a_3) a_3 (a_3 a_4) a_4 (a_4 a_5) a_5 (a_5 a_6) a_6 (a_6 a_0) a_0 (a_0 a_1) a_1 \] (1.5)

Here \( d(a_0) = 6, \ d(a_i) = 2 \) for \( i = 1, 2, \ldots, 6 \).

2. MAIN RESULTS

Here we prove some results relating to open pyramid graph and BCK algebras.

**Theorem (2.1)** - For a given open pyramid graph of order \( n + 1 \) where \( n = 2m \) there exists a BCK – algebra on a set of \( n + 1 \) elements such that the simple graph associated with mutually disjoint elements coincide with the given open pyramid graph.

**Proof** - Let \( a_0 \) be the vertex and \( a_1, a_2, \ldots, a_n \) be base points of an open pyramid graph of order \( n + 1 \). Let \( E = \{ a_0, a_1, a_2, \ldots, a_n \} \).

Then as given in definition (1.7) edges of open pyramid graph are \( a_0 a_1, a_0 a_2, \ldots, a_0 a_n, a_1 a_2, a_2 a_3, \ldots, a_{n-1} a_n \).

We consider a binary operation \("\ast\) in \( E \) such that \( a_0 \) is taken as zero element and an edge connects two distinct points of the open pyramid graph if the points are mutually disjoint. This means that pairs \( \{ a_0, a_1 \}, \{ a_0, a_2 \}, \ldots, \{ a_{n-1}, a_n \} \) contain mutually disjoint elements. So binary operation \("\ast\) must satisfy
\[ a_0 \ast a_1 = a_0 \ast a_2 = \ldots = a_0 \ast a_n = a_0 \] (2.1)
\[ a_1 \ast a_0 = a_1 \ast a_2 = \ldots = a_1 \ast a_n = a_0 \] (2.2)
\[ a_2 \ast a_1 = a_2 \ast a_3 = \ldots = a_{n-1} \ast a_n = a_0 \] (2.3)

We also assume that \( a_i \ast a_i = a_0 \) for \( i = 0, 1, 2, \ldots, n \).

For pairs \( \{ a_2, a_3 \}, \{ a_4, a_5 \}, \ldots, \{ a_{n-2}, a_{n-1} \} \) which are not connected by an edge, we define
\[ a_i \ast a_j = a_0 \text{ and } a_j \ast a_i = a_0, i < j \] (2.4)
or \[ a_i \ast a_j = a_0 \text{ and } a_j \ast a_i = a_i, i < j \] (2.5)

In other words, the elements of these pairs are taken as semi mutually disjoint. Now using theorem (1.3) and corollary (1.4) we see that \( (E, \ast, a_0) \) is a BCK – algebra.

If two points of \( E \) are connected by an edge iff they are mutually disjoint then the simple graph so obtained coincides with given open pyramid graph.

Hence the result.
Theorem (2.2): For a given open pyramid graph of order \( n + 1 \) where \( n = 2m \) there exists a BCK-algebra on a set of \( n + 1 \) elements such that the simple graph associated with semi-mutually disjoint elements coincide with the given open pyramid graph.

**Proof:** Let \( a_0 \) be the vertex and \( a_1, a_2, \ldots, a_n \) be base points of an open pyramid graph of order \( n + 1 \). Let \( E = \{ a_0, a_1, a_2, \ldots, a_n \} \).

Then as given in definition (1.7) edges of open pyramid graph are \( a_0, a_1, \ldots, a_0 a_n, a_1 a_2, a_3 a_4, \ldots, a_{n-1} a_n \).

We consider a binary operation ‘**’ in \( E \) such that \( a_0 \) is taken as zero element and an edge connects two distinct points of the open pyramid graph iff the points are semi-mutually disjoint. This means that pairs \( \{a_2, a_1\}, \{a_0, a_2\}, \ldots, \{a_0, a_n\}, \{a_1, a_2\}, \ldots, \{a_{n-1}, a_n\} \)

contain semi-mutually disjoint elements. So binary operation ‘**’ must satisfy

\[
a_0 * a_1 = a_0, a_1 * a_0 = a_1; a_0 * a_2 = a_0, a_2 * a_0 = a_2, \ldots, \ldots a_0 * a_n-1 = a_0, a_n-1 * a_0 = a_n-1, a_0 * a_n = a_0, a_n * a_0 = a_n. \tag{2.6}
\]

\[
a_1 * a_2 = a_1, a_2 * a_1 = a_0; a_1 * a_3 = a_3, a_3 * a_1 = a_3 \ldots, \ldots a_n-1 * a_n = a_n-1, a_n * a_n-1 = a_0 \tag{2.7}
\]

We also assume \( a_i * a_i = 0 \) for \( i = 0, 1, 2, \ldots, n \).

For pairs \( \{a_2, a_3\}, \{a_4, a_5\}, \ldots, \{a_{n-2}, a_{n-1}\} \) which are not connected by an edge, we define

\[
a_2 * a_3 = a_2, a_3 * a_2 = a_3, \ldots, a_{n-2} * a_{n-1} = a_{n-2}, a_{n-1} * a_n = a_{n-1} \tag{2.8}
\]

In other words, the elements of these pairs are taken as mutually disjoint. Using corollary (1.4), we see that \((E, *, a_0)\) is a BCK-algebra. Further, if two points of \( E \) are connected by an edge iff they are semi-mutually disjoint then the simple graph so obtained coincides with the given open pyramid graph.

Hence the result.

We also have the following useful result.

**Theorem (2.3):** Every open pyramid graph with vertex \( a_0 \) and base points \( a_1, a_2, \ldots, a_n \) \( n = 2m \) has an Euler circuit.

**Proof:** Suppose that \( P(m) \) stands for the statement given in the theorem (2.3). From examples (1.6) and (1.8) we see that \( P(1), P(2), \) and \( P(3) \) are satisfied. We assume that \( P(m – 1) \) is true. Then as explained definition (1.7) we have Euler circuit as

\[
a_1 (a_1 a_2) a_2 (a_2 a_3) a_3 (a_3 a_4) a_4 \ldots (a_{2m-2} (a_{2m-2} a_{2m-1}) a_{2m-1} (a_{2m-1} a_0) a_0 (a_0 a_1) a_1 \tag{2.9}
\]

Now we introduce two points \( a_{2m-2} \) and \( a_{2m} \).

These points are connected with \( a_0 \) as \( (a_{2m-1} a_0) \) and \( (a_{2m} a_0) \). Also \( a_{2m-1} \) and \( a_{2m} \) are connected by \( (a_{2m-1} a_{2m}) \).

Now we change the circuit of (2.9) by omitting \( a_0 (a_0 a_1) a_1 \) and introducing \( a_0 (a_0 a_{2m-1}) a_{2m-1} (a_{2m-1} a_{2m}) a_{2m} (a_{2m} a_0) a_0 (a_0 a_1) a_1 \).

This gives a circuit in the open pyramid graph with vertex \( a_0 \) and base points \( a_1, \ldots, a_n \) where \( n = 2m \). In other words \( P(m) \) is satisfied.

So using Principles of Mathematical induction we see that \( P(m) \) is satisfied for all positive integers \( m \).

**REFERENCES**


