Dominator coloring of certain graphs

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ABSTRACT

A graph has a dominator coloring if it has a proper coloring in which each vertex of the graph dominates every vertex of some color class. The dominator chromatic number \( \chi_d(G) \) is the minimum number of colors required for a dominator coloring of \( G \). In this paper the dominator chromatic number for comb graph, jewel graph, flower graph is studied and also the dominator coloring of corona graph of path with triangular snake graph and double triangular snake graph is studied.

Keywords: Comb Graph, Flower Graph, Jewel Graph, Corona Product Graph, Dominator Coloring, Dominator Chromatic Number

1. INTRODUCTION

A dominating set \( S \) is a subset of the vertices in a graph such that every vertex in the graph either belongs to \( S \) or has a neighbor in \( S \). The domination number, \( \gamma(G) \), is the order of a minimum dominating set.

A proper coloring of a graph \( G = (V(G), E(G)) \) is an assignment of colors to the vertices of the graph, such that any two adjacent vertices have different colors. The chromatic number is the minimum number of colors needed in a proper coloring of graph.

One of the important areas in graph theory is graph coloring which is used in many applications like Map coloring, Sudoku, Register Allocations, Mobile radio frequency assignment and Making schedule or time table and it has played a major role in the development of graph theory and, more generally, discrete mathematics and combinatorial optimization.

A dominator coloring of a graph \( G \) is a proper coloring in which each vertex of the graph dominates every vertex of some color class. The dominator chromatic number \( \chi_d(G) \) is the minimum number of color classes in a dominator coloring of a graph \( G \). This concept was introduced by Raluca Michelle Gera in 2006 [4].

2. DOMINATOR COLORING OF COMB, JEWEL AND FLOWER GRAPHS

In this section we have obtained the dominator chromatic number of comb graph \( P_n \), Jewel graph \( J_n \), Flower graph \( F_n \).

2.1 Dominator coloring of comb graph

Definition 2.1.1. Let \( P_n \) be a path graph with \( n \) vertices. The comb graph is defined as \( P_n \odot K_1 \). It has \( 2n \) vertices and \( 2n-1 \) edges.

Theorem 2.1.1. For a comb graph \( P_n \), with \( 2n \) vertices, \( \chi_d(P_n) = n + 1 \).

Proof. Let \( P_n \) be a comb graph with \( 2n \) vertices. Label the vertices of \( P_n \) as follows:

Path vertices are labelled as \( \{v_1, v_2, ..., v_n\} \) and pendant vertices are labelled as \( \{v_{n+1}, v_{n+2}, ..., v_{2n}\} \) where
\[ V(G) = \{v_1, v_2, \ldots, v_n\} \cup \{v_{n+1}, v_{n+2}, \ldots, v_{2n}\} \] 

Now choose the vertices of path given by \(\{v_1, v_2, \ldots, v_n\}\). It yields the minimum dominating set for \(P_n^+\).

We now claim the \(\chi_d(P_n^+) = n + 1\)

Suppose \(\chi_d(P_n^+) \neq n + 1\) Let \(\chi_d(P_n^+) \leq n\).

Assume \(\chi_d(P_n^+) = n\). Define a function \(C: V(G) \to \{1, 2, \ldots, n\}\) such that the path vertices will receive \(\{1, 2, \ldots, n\}\) colors and the vertex \(v_{n+1}\) will not receive the color 1 which contradicts the coloring property of \(G\) and the vertex \(v_{n+1}\) will not colored with colors \(\{2, 3, \ldots, n\}\) that will contradict the dominator coloring condition i.e. each vertex of the graph dominates every vertex of some color class.

Hence \(\chi_d(P_n^+) \neq n\).

Thus, \(\chi_d(P_n^+) \geq n + 1\) Let us assign the colors \(c_1, c_2, \ldots, c_n\) to the vertices \(\{v_1, v_2, \ldots, v_n\}\) and assign the color \(c_{n+1}, \ldots\) to the remaining non-adjacent vertices of the graph. By the definition of dominator coloring all the vertices in the dominating set of \(P_n^+\) dominates its own color class itself and other vertices also dominates all the other color classes (refer to Illustration 2.1.1).

Hence \(\chi_d(P_n^+) = n + 1\).

**Illustration 2.1.1.** Dominator coloring of comb graph \(P_n^+\).

![Fig. 2.1: Dominator coloring of \(P_n^+\)](image)

The color classes are \(C_1 = \{v_1\}, C_2 = \{v_2\}, C_3 = \{v_3\}, C_4 = \{v_4\}, C_5 = \{v_5, v_6, v_7, v_8\}\).

The vertices \(\{v_1, v_2, v_3, v_4\}\) will dominates its own color class and the vertex \(v_5\) will dominate the color class 1, the vertex \(v_6\) will dominate the color class 2, the vertex \(v_7\) will dominate the color class 3, and the vertex \(v_8\) will dominate the color class 4 and hence by the definition of dominator coloring, each vertex of the graph dominates every vertex of some color class.

Therefore, the dominator chromatic number for comb graph \(P_n^+\) is 5.

### 2.2 Dominator coloring of jewel graph

**Definition 2.2.1.** The jewel graph \(J_n\) is a graph with vertex set \(V(J_n) = \{u, v, x, y, u_1 : 1 \leq i \leq n\}\) and an edge set \(E(J_n) = \{ux, vy, xy, yv, uu_1, vuv : 1 \leq i \leq n\}\).

**Theorem 2.2.1.** For a jewel graph \(J_n\), \(\chi_d(J_n) = 3\).

Proof. Let \(J_n\) be a Jewel graph with \(n\) vertices. Let \(V = \{u, v, x, y, u_1, u_2, \ldots, u_n\}\) be the vertex set. We only need two vertices to form a minimum dominating set. From Figure 2.2 we know that \(D = \{u, v\}\) form a minimum dominating set.

We now claim the \(\chi_d(J_n) = 3\)

Suppose \(\chi_d(J_n) \neq 3\) and let \(\chi_d(J_n) \leq 2\).

Assume \(\chi_d(J_n) = 2\) Define a function \(C: V(J_n) \to \{1, 2\}\) such that the dominating set \(D = \{u, v\}\) vertices will receive color 1 and the vertices \(\{x, u_1, u_2, \ldots, u_n\}\) will receive the color 2. Since the vertex \(y\) cannot be colored with colors 1 or 2 which contradicts the coloring property of \(G\).

Thus \(\chi_d(J_n) \neq 3\) (refer to the Illustration 2.2.1). Hence \(\chi_d(J_n) = 3\).
The color classes are \( C_1 = \{ u, v \} \), \( C_2 = \{ x, u_1, u_2, u_3, u_4 \} \), \( C_3 = \{ y \} \).

The vertices \( \{ u_1, u_2, u_3, u_4, x \} \) will dominates the color class 1 and the vertices \( \{ u, v \} \) will dominate the color class 2 and 3, the vertex \( y \) will dominate the color class 1 and hence by the definition of dominator coloring, each vertex of the graph dominates every vertex of some color class. Therefore, the dominator chromatic number for jewel graph \( J_4 \) is 3.

2.3 Dominator coloring of flower graph

**Definition 2.3.1.** A Flower graph \( F_n \) is a graph obtained from a Helm by joining each pendent vertex to the central vertex of helm.

**Theorem 2.3.1.** For any flower graph \( F_n \), \( \chi_d(F_n) = 3 \), \( n \geq 4 \)

Proof. Let \( V(F_n) = \{ v, v_1, u_i : 1 \leq i \leq n \} \) and \( E(F_n) = \{ vv_1, vjv_{j+1}, vnv_{i-1}, vnv_i : 1 \leq i \leq n, 1 \leq j \leq n - 1 \} \). First select the central vertex \( v \) of \( F_n \), this will dominate all other vertices of \( F_n \).

We now claim the \( \chi_d(F_n) = 2 \)

Suppose \( \chi_d(F_n) \neq 2 \) let \( \chi_d(F_n) \leq 1 \).

Assume \( \chi_d(F_n) = 2 \). Define a function \( C : V(F_n) \to \{ 1, 2 \} \) such that the central vertex \( v \) will receive color 1 and so that all the other vertices cannot receive the color 1 since it is adjacent with all the other vertices. The vertex \( v_1 \) will receive the color 2 so that the vertex \( u_1 \) cannot be colored using 1 or 2 which contradicts the coloring property of \( G \).

Thus \( \chi_d(F_n) \neq 2 \) (refer to the Illustration 2.3.1). Hence \( \chi_d(F_n) = 2 \).

**Illustration 2.3.1.** Dominator coloring of flower graph \( F_4 \).

The color classes are \( C_1 = \{ v \} \), \( C_2 = \{ u_1, u_2, v_2, v_4 \} \), \( C_3 = \{ u_2, u_4, v_1, v_3 \} \).

The vertices \( \{ u_1, u_2, u_3, u_4, v_1, v_2, v_3, v_4 \} \) will dominates the color class 1 and the vertex \( v \) will dominate its own color class and hence by the definition of dominator coloring, each vertex of the graph dominates every vertex of some color class. Therefore, the dominator chromatic number for flower graph \( F_4 \) is 3.

3. DOMINATOR COLORING ON CORONA GRAPH OF PATH WITH SOME GRAPHS

In this section we obtain the dominator chromatic number on corona graph of path with triangular snake graph \( T_n \), corona graph of path with double triangular snake graph \( D(T_n) \).
Definition 3.1. Let $G_1$ and $G_2$ be two graphs on disjoint sets of $n$ and $m$ vertices, respectively. The corona graph $G_1 \circ G_2$ of $G_1$ and $G_2$ is defined as the graph obtained by taking one copy of $G_1$ and $n$ copies of $G_2$, and then joining the $i$th vertex of $G_1$ to every vertex in the $i$th copy of $G_2$.

Example 3.1. Let $G_1 = C_4$, the cycle of order 4 and $G_2 = K_2$. The corona $G_1 \circ G_2$ of graphs of $G_1$ and $G_2$ are shown in Figures 3.1 and 3.2.

![Fig. 3.1: Graphs $G_1$ and $G_2$](image1)

![Fig. 3.2: Corona graph $G_1 \circ G_2$](image2)

Definition 3.2. A walk in which no vertex appears more than once is called a simple path or a path, that is each vertex in the path is distinct. Path on $n$ vertices is $P_n$.

3.1 Dominator chromatic number on corona graph of path with triangular snake graph

Definition 3.1.1. A triangular snake graph $T_n$ is obtained from the path $P_n$ by replacing each edge by a triangle $C_3$.

Theorem 3.1.1. Let $P_n$ and $T_n$ denote the path and triangular snake graph with $n$ vertices then $\chi_d(P_n \circ T_n) = n + 3, n \geq 2$.

Proof. Let $V(P_n) = \{v_i : 1 \leq i \leq n\}$ and $V(T_n) = \{u_i : 1 \leq i \leq n\}$.

Let $V(P_n \circ T_n) = \{v_i : 1 \leq i \leq n\} \cup \{u_{ij} : 1 \leq i, j \leq n\}$.

By the definition of corona graph, each vertex of $P_n$ is adjacent to every vertex of a copy of $T_n$ (i.e., every vertex $v_i \in V(P_n)$ is vertex from the set $\{u_{ij} : 1 \leq j \leq n\}$). Now choose the vertices of the path $\{v_1, v_2, \ldots, v_n\}$. It yields a minimum dominating set $D$.

We now claim that $\chi_d(P_n \circ T_n) = n + 3$. Suppose $\chi_d(P_n \circ T_n) = n + 2$, let $\chi_d(P_n \circ T_n) \leq n + 2$.

Assume that $\chi_d(P_n \circ T_n) = n + 2$. Define a function $C : V(P_n \circ T_n) \rightarrow \{1, 2, \ldots, n+2\}$ such that the dominating set $D$ will receive the color $\{1, 2, \ldots, n\}$. Now the vertex $u_{i1}$ will receive the color $n + 1$ and $u_{3i}$ will receive the color $n + 3$ but $u_{3i}$ cannot be colored using the color $n + 1$ or $n + 2$ or 1 which contradicts the coloring property and also $u_{3i}$ cannot be colored using $\{1, 2, \ldots, n\}$ which will not satisfy the dominator coloring condition, that is each vertex of the graph dominates every vertex of some color class.

Thus $\chi_d(P_n \circ T_n) = n + 2$ (refer to the Illustration 3.1.1). Hence $\chi_d(P_n \circ T_n) = n + 3$.

Illustration 3.1.1. Dominator chromatic number on corona graph of path $P_4$ with triangular snake graph $T_4$.

![Fig. 3.3: Dominator coloring of $P_4 \circ T_4$](image3)

The color classes are $C_1 = \{v_1\}, C_2 = \{v_2\}, C_3 = \{v_3\}, C_4 = \{v_4\}, C_5 = \{u_{15}, u_{16}, u_{17}, u_{25}, u_{26}, u_{27}, u_{35}, u_{36}, u_{37}, u_{45}, u_{46}, u_{47}\}, C_6 = \{u_{11}, u_{13}, u_{31}, u_{33}, u_{21}, u_{23}, u_{31}, u_{33}, u_{41}, u_{43}\}, C_7 = \{u_{12}, u_{14}, u_{22}, u_{24}, u_{32}, u_{34}, u_{42}, u_{44}\}$.

The vertices $\{v_1, v_2, v_3, v_4\}$ will dominates its own color class 1, 2, 3 and 4 respectively. The vertices $\{u_{15}, u_{12}, u_{13}, u_{14}, u_{15}, u_{16}, u_{17}\}$ will dominates the color class 1. The vertices $\{u_{25}, u_{22}, u_{23}, u_{24}, u_{25}, u_{26}, u_{27}\}$ will dominates the color class 2. The vertices $\{u_{35}, u_{32}, u_{33}, u_{34}, u_{35}, u_{36}, u_{37}\}$ will dominates the color class 3 and the vertices $\{u_{45}, u_{42}, u_{43}, u_{44}, u_{45}, u_{46}, u_{47}\}$ will dominates the color class 4. Hence by the definition of dominator coloring, each vertex of the graph dominates every vertex of some color class.
Therefore, the dominator chromatic number for corona graph of path \( P_n \) with triangular snake graph \( T_s \) is 7 (i.e.) \( X_d(P_n \circ T_s) = 7 \).

3.2. Dominator chromatic number on corona graph of path with double triangular snake graph

**Definition 3.2.1.** A double triangular snake graph \( D(T_n) \) is obtained from the path \( u_1u_2 \ldots u_n \) by joining \( u_i, u_{i+1} \) with two new vertices \( v_i \) and \( w_i \) for \( 1 \leq i \leq n-1 \).

**Theorem 3.2.1.** Let \( P_n \) and \( D(T_n) \) denote the path and double triangular snake graph with \( n \) vertices then
\[
X_d(P_n \circ D(T_n)) = n + 3
\]

Proof. Let \( V(P_n) = \{v_i : 1 \leq i \leq n \} \) and \( V(D(T_n)) = \{u_i : 1 \leq i \leq n \} \) and \( V(P_n \circ D(T_n)) = \{v_i : 1 \leq i \leq n \} \cup \{u_{ij} : 1 \leq i, j \leq n \} \).

By the definition of corona graph, each vertex of \( P_n \) is adjacent to every vertex of a copy of \( D(T_n) \), that is every vertex \( v_i \in V(P_n) \) is adjacent to every vertex from the set \( \{u_i : 1 \leq j \leq n \} \). Now choose the vertices of the path \( \{v_1, v_2, \ldots, v_n\} \). It yields a minimum dominating set \( D \).

We now claim the \( X_d(P_n \circ D(T_n)) = n + 3 \).

Suppose \( X_d(P_n \circ D(T_n)) \neq n + 3 \) let \( X_d(P_n \circ D(T_n)) \leq n + 2 \).

Assume that \( X_d(P_n \circ D(T_n)) = n + 2 \). Define a function \( C : V(P_n \circ D(T_n)) \rightarrow \{1, 2, \ldots, n + 2\} \) such that the dominating set \( D \) will receive the color \( \{1, 2, \ldots, n\} \). Now the vertex \( u_1 \) will receive the color \( n + 1 \) and \( u_2 \) will receive the color \( n + 3 \) but \( u_3 \) cannot be colored using the color \( n + 1 \) or \( n + 2 \) or 1 which contradicts the coloring property and also \( u_3 \) cannot be colored using \( \{1, 2, \ldots, n\} \) which will not satisfy the dominator coloring condition (i.e.) each vertex of the graph dominates every vertex of some color class.

Thus \( X_d(P_n \circ D(T_n)) \neq n + 2 \) (refer to the Illustration 3.2.1.). Hence \( X_d(P_n \circ D(T_n)) = n + 3 \).

**Illustration 3.2.1.** Dominator chromatic number on corona graph of path \( P_n \) with double triangular snake graph \( D(T_s) \).

![Fig. 3.4: Dominator coloring of \( P_n \circ D(T_s) \)](image)

The color classes are \( C_1 = \{v_1\}, C_2 = \{v_2\}, C_3 = \{v_3\} \), \( C_4 = \{v_4\}, C_5 = \{u_{15}, u_{16}, u_{17}, u_{18}, u_{19}, u_{110}, u_{25}, u_{26}, u_{27}, u_{28}, u_{29}, u_{30}, u_{31}, u_{32}, u_{33}, u_{34}, u_{35}, u_{36}, u_{37}, u_{38}, u_{39}, u_{40}, u_{41}, u_{42}, u_{43}, u_{44}\} \), \( C_6 = \{u_{11}, u_{12}, u_{13}, u_{14}, u_{15}, u_{16}, u_{17}, u_{18}, u_{19}, u_{20}\} \) will dominate the color class 1. The vertices \( \{u_{11}, u_{12}, u_{13}, u_{14}, u_{15}, u_{16}, u_{17}, u_{18}, u_{19}, u_{20}\} \) will dominate the color class 2. The vertices \( \{u_{21}, u_{22}, u_{23}, u_{24}, u_{25}, u_{26}, u_{27}, u_{28}, u_{29}, u_{30}\} \) will dominate the color class 3 and the vertices \( \{u_{41}, u_{42}, u_{43}, u_{44}, u_{45}, u_{46}, u_{47}, u_{48}, u_{49}, u_{50}\} \) will dominate the color class 4. Hence by the definition of dominator coloring, each vertex of the graph dominates every vertex of some color class. Therefore, the dominator chromatic number for corona graph of path \( P_n \) with double triangular snake graph \( D(T_s) \) is 7 (i.e.) \( X_d(P_n \circ D(T_s)) = 7 \).

4. CONCLUSION

In this paper we obtain the dominator chromatic number of comb graph, jewel graph and flower graph and also the dominator chromatic number of corona graph of path with triangular snake graph and corona graph of path with double triangular snake graph. This paper can be further extended by identifying some families of graph and corona graph of path with any graphs.
5. REFERENCES


