



# INTERNATIONAL JOURNAL OF ADVANCE RESEARCH, IDEAS AND INNOVATIONS IN TECHNOLOGY

ISSN: 2454-132X

Impact Factor: 6.078

(Volume 7, Issue 3 - V7I3-1502)

Available online at: <https://www.ijariit.com>

## Sequences of Gaussian Diophantine 3-Tuples

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### ABSTRACT

*This paper deals with the study of constructing sequences of diophantine triples  $(a, b, c)$  through Gaussian integers such that the product of any two elements of the set added by a non-zero integer or a polynomial with integer coefficients or Gaussian integer is a perfect square.*

**Keywords:** Diophantine 3-tuple, Gaussian 3-tuple

**2010 Mathematics Subject Classification:** 11D99

### 1. INTRODUCTION

A theory that can be explained in a regular and systematic way is a pattern. The essence of mathematical calculations is represented by numbers and the theory of numbers can be taught as a set of patterns as it can be explained in a regular and systematic way. It is to be noted here that the number pattern is a sequence of numbers establishing a common relationship between them. Numbers exhibit fascinating, beautiful and curious varieties of patterns, namely, polygonal numbers, Fibonacci numbers, Lucas numbers, Ramanujan numbers, Kynea numbers, Jacobsthal numbers and so on. The problem of constructing the sets in distinct integers with property that the product of any two of its members is one less than a square has a very long history and such sets were studied by Diophantus. In particular, if  $S$  is a set of three non-zero distinct integers  $a_1, a_2, a_3$  such that the product of any two members of the set with the addition of  $n$  (a non-zero integer or a polynomial with integer coefficients) is a perfect square, then  $S$  is said to be Diophantine 3-tuple with property  $D(n)$ . In this context one may refer [1-16].

This paper concerns with the analogous problems for Gaussian integers. A Gaussian integer is a complex number of the form  $a + ib, i = \sqrt{-1}$ , where real and imaginary parts are non-zero integers. A set of three distinct non-zero Gaussian integers is said to be a Gaussian Diophantine 3-tuple with property  $D(n)$  if the product of any two members of set with addition of  $n$  (a non-zero integer or a polynomial with integer coefficients or Gaussian integer) is a perfect square. The construction of sequences of Gaussian Diophantine 3-tuples with suitable properties is illustrated in this paper.

### 2. METHOD OF ANALYSIS

**Sequence: 1**

Let  $a = \alpha + ik\beta$ ,  $c_0 = \alpha + i(k+4)\beta$

It is observed that

$$ac_0 - 4\beta^2 = [\alpha + i(k+2)\beta]^2$$

Therefore, the pair  $(a, c_0)$  represents diophantine 2-tuple with the property  $D(-4\beta^2)$ .

Let  $c_1$  be any non-zero polynomial such that

$$ac_1 - 4\beta^2 = p^2 \tag{1}$$

$$c_0c_1 - 4\beta^2 = q^2 \tag{2}$$

Eliminating  $c_1$  between (1) and (2), we have

$$c_0p^2 - aq^2 = (a - c_0)4\beta^2 \tag{3}$$

Introducing the linear transformations

$$p = X + aT, \quad q = X + c_0T \tag{4}$$

in (3) and simplifying we get

$$X^2 = ac_0 T^2 - 4\beta^2$$

which is satisfied by  $T = 1, X = \alpha + i(k + 2)\beta$

In view of (4) and (1), it is seen that

$$c_1 = 4\alpha + i\beta(4k + 8)$$

Note that  $(a, c_0, c_1)$  represents diophantine 3-tuple with property  $D(-4\beta^2)$

Taking  $(a, c_1)$  and employing the above procedure, it is seen that the triple  $(a, c_1, c_2)$  where

$$c_2 = 9\alpha + i\beta(9k + 12)$$

exhibits diophantine 3-tuple with property  $D(-4\beta^2)$

Taking  $(a, c_2)$  and employing the above procedure, it is seen that the triple  $(a, c_2, c_3)$  where

$$c_3 = 16\alpha + i\beta(16k + 16)$$

exhibits diophantine 3-tuple with property  $D(-4\beta^2)$

Taking  $(a, c_3)$  and employing the above procedure, it is seen that the triple  $(a, c_3, c_4)$  where

$$c_4 = 25\alpha + i\beta(25k + 20)$$

exhibits diophantine 3-tuple with property  $D(-4\beta^2)$

The repetition of the above process leads to the generation of sequence of diophantine 3-tuples whose general form is given by  $(a, c_{\partial-1}, c_{\partial})$  where

$$c_{\partial-1} = \partial^2\alpha + i\beta(\partial^2k + 4\partial), \quad \partial = 1, 2, 3, \dots$$

**Table 1: Numerical Examples with property  $D(-4\beta^2)$**

<b>k</b>	<b><math>\alpha</math></b>	<b><math>\beta</math></b>	<b><math>(a, c_0, c_1)</math></b>	<b><math>(a, c_1, c_2)</math></b>	<b><math>(a, c_2, c_3)</math></b>
1	3	2	$(3 + i2, 3 + i10, 12 + i24)$	$(3 + i2, 12 + i24, 27 + i42)$	$(3 + i2, 27 + i42, 48 + i64)$
2	4	3	$(4 + i6, 4 + i18, 16 + i48)$	$(4 + i6, 16 + i48, 36 + i90)$	$(4 + i6, 36 + i90, 64 + i144)$
3	5	4	$(5 + i12, 5 + i28, 20 + i80)$	$(5 + i12, 20 + i80, 45 + i156)$	$(5 + i12, 45 + i156, 80 + i256)$
4	6	5	$(6 + i20, 6 + i40, 24 + i120)$	$(6 + i20, 24 + i120, 54 + i240)$	$(6 + i20, 54 + i240, 96 + i400)$
5	7	6	$(7 + i30, 7 + i54, 28 + i168)$	$(7 + i30, 28 + i168, 63 + i342)$	$(7 + i30, 63 + i342, 112 + i576)$

**Sequence: 2**

Let  $a = \alpha + i(k + 4)\beta, \quad c_0 = \alpha + ik\beta$

It is observed that

$$ac_0 - 4\beta^2 = [\alpha + i(k + 2)\beta]^2$$

Therefore, the pair  $(a, c_0)$  represents diophantine 2-tuple with the property  $D(-4\beta^2)$ .

Let  $c_1$  be any non-zero polynomial such that

$$ac_1 - 4\beta^2 = p^2 \tag{5}$$

$$c_0c_1 - 4\beta^2 = q^2 \tag{6}$$

Eliminating  $c_1$  between (5) and (6), we have

$$c_0p^2 - aq^2 = (a - c_0)4\beta^2 \tag{7}$$

Introducing the linear transformations

$$p = X + aT, \quad q = X + c_0T \tag{8}$$

in (7) and simplifying we get

$$X^2 = ac_0 T^2 - 4\beta^2$$

which is satisfied by  $T = 1, X = \alpha + i(k + 2)\beta$

In view of (8) and (5), it is seen that

$$c_1 = 4\alpha + i\beta(4k + 8)$$

Note that  $(a, c_0, c_1)$  represents diophantine 3-tuple with property  $D(-4\beta^2)$

Taking  $(a, c_1)$  and employing the above procedure, it is seen that the triple  $(a, c_1, c_2)$  where

$$c_2 = 9\alpha + i\beta(9k + 24)$$

exhibits diophantine 3-tuple with property  $D(-4\beta^2)$

Taking  $(a, c_2)$  and employing the above procedure, it is seen that the triple  $(a, c_2, c_3)$  where

$$c_3 = 16\alpha + i\beta(16k + 48)$$

exhibits diophantine 3-tuple with property  $D(-4\beta^2)$

Taking  $(a, c_3)$  and employing the above procedure, it is seen that the triple  $(a, c_3, c_4)$  where

$$c_4 = 25\alpha + i\beta(25k + 80)$$

exhibits diophantine 3-tuple with property  $D(-4\beta^2)$

The repetition of the above process leads to the generation of sequence of diophantine 3-tuples whose general form is given by  $(a, c_{\partial-1}, c_{\partial})$  where

$$c_{\partial-1} = \partial^2\alpha + i\beta(\partial^2k + 4\partial(\partial - 1)), \partial = 1, 2, 3, \dots$$

**Table 2: Numerical Examples with property  $D(-4\beta^2)$**

k	$\alpha$	$\beta$	$(a, c_0, c_1)$	$(a, c_1, c_2)$	$(a, c_2, c_3)$
1	3	2	$(3 + i10, 3 + i2, 12 + i24)$	$(3 + i10, 12 + i24, 27 + i66)$	$(3 + i10, 27 + i66, 48 + i128)$
2	4	3	$(4 + i18, 4 + i6, 16 + i48)$	$(4 + i18, 16 + i48, 36 + i126)$	$(4 + i18, 36 + i126, 64 + i240)$
3	5	4	$(5 + i28, 5 + i12, 20 + i80)$	$(5 + i28, 20 + i80, 45 + i204)$	$(5 + i28, 45 + i204, 80 + i384)$
4	6	5	$(6 + i40, 6 + i20, 24 + i120)$	$(6 + i40, 24 + i120, 54 + i300)$	$(6 + i40, 54 + i300, 96 + i560)$
5	7	6	$(7 + i54, 7 + i30, 28 + i168)$	$(7 + i54, 28 + i168, 63 + i414)$	$(7 + i54, 63 + i414, 112 + i768)$

**Sequence: 3**

Let  $a = \alpha + ik\beta$ ,  $c_0 = \alpha + i\beta(k + 3)$

It is observed that

$$ac_0 + i\alpha\beta - (k + 4)\beta^2 = [\alpha + i(k + 2)\beta]^2$$

Therefore, the pair  $(a, c_0)$  represents diophantine 2-tuple with the property  $D(i\alpha\beta - (k + 4)\beta^2)$ .

Let  $c_1$  be any non-zero polynomial such that

$$ac_1 + i\alpha\beta - (k + 4)\beta^2 = p^2 \tag{9}$$

$$c_0c_1 + \alpha\beta - (k + 4)\beta^2 = q^2 \tag{10}$$

Eliminating  $c_1$  between (9) and (10), we have

$$c_0p^2 - aq^2 = (c_0 - a)(i\alpha\beta - (k + 4)\beta^2) \tag{11}$$

Introducing the linear transformations

$$p = X + aT, \quad q = X + c_0T \tag{12}$$

in (11) and simplifying we get

$$X^2 = ac_0T^2 + i\alpha\beta - (k + 4)\beta^2$$

which is satisfied by  $T = 1$ ,  $X = \alpha + i(k + 2)\beta$

In view of (12) and (9), it is seen that

$$c_1 = 4\alpha + i\beta(4k + 7)$$

Note that  $(a, c_0, c_1)$  represents diophantine 3-tuple with property  $D(i\alpha\beta - (k + 4)\beta^2)$

Taking  $(a, c_1)$  and employing the above procedure, it is seen that the triple  $(a, c_1, c_2)$  where

$$c_2 = 9\alpha + i\beta(9k + 11)$$

exhibits diophantine 3-tuple with property  $D(i\alpha\beta - (k + 4)\beta^2)$

Taking  $(a, c_2)$  and employing the above procedure, it is seen that the triple  $(a, c_2, c_3)$  where

$$c_3 = 16\alpha + i\beta(16k + 15)$$

exhibits diophantine 3-tuple with property  $D(i\alpha\beta - (k + 4)\beta^2)$

Taking  $(a, c_3)$  and employing the above procedure, it is seen that the triple  $(a, c_3, c_4)$  where

$$c_4 = 25\alpha + i\beta(25k + 19)$$

exhibits diophantine 3-tuple with property  $D(i\alpha\beta - (k + 4)\beta^2)$

The repetition of the above process leads to the generation of sequence of diophantine 3-tuples whose general form is given by  $(a, c_{\partial-1}, c_{\partial})$  where

$$c_{\partial-1} = \partial^2\alpha + i\beta(\partial^2k + 4\partial - 1), \partial = 1, 2, 3, \dots$$

**Table 3: Numerical Examples with property  $D(i\alpha\beta - (k + 4)\beta^2)$**

k	$\alpha$	$\beta$	$(a, c_0, c_1)$	$(a, c_1, c_2)$	$(a, c_2, c_3)$
1	3	2	$(3 + i2, 3 + i8, 12 + i22)$	$(3 + i2, 12 + i22, 27 + i40)$	$(3 + i2, 27 + i40, 48 + i62)$
2	4	3	$(4 + i6, 4 + i15, 16 + i45)$	$(4 + i6, 16 + i45, 36 + i87)$	$(4 + i6, 36 + i87, 64 + i141)$
3	5	4	$(5 + i12, 5 + i24, 20 + i76)$	$(5 + i12, 20 + i76, 45 + i152)$	$(5 + i12, 45 + i152, 80 + i252)$
4	6	5	$(6 + i20, 6 + i35, 24 + i115)$	$(6 + i20, 24 + i115, 54 + i235)$	$(6 + i20, 54 + i235, 96 + i395)$
5	7	6	$(7 + i30, 7 + i48, 28 + i162)$	$(7 + i30, 28 + i162, 63 + i336)$	$(7 + i30, 63 + i336, 112 + i570)$

**Sequence: 4**

Let  $a = \alpha + i\beta(k + 3), c_0 = \alpha + ik\beta$

It is observed that

$$ac_0 + i\alpha\beta - (k + 4)\beta^2 = [\alpha + i(k + 2)\beta]^2$$

Therefore, the pair  $(a, c_0)$  represents diophantine 2-tuple with the property  $D(i\alpha\beta - (k + 4)\beta^2)$ .

Let  $c_1$  be any non-zero polynomial such that

$$ac_1 + i\alpha\beta - (k + 4)\beta^2 = p^2 \tag{13}$$

$$c_0c_1 + \alpha\beta - (k + 4)\beta^2 = q^2 \tag{14}$$

Eliminating  $c_1$  between (13) and (14), we have

$$c_0p^2 - aq^2 = (c_0 - a)(i\alpha\beta - (k + 4)\beta^2) \tag{15}$$

Introducing the linear transformations

$$p = X + aT, q = X + c_0T \tag{16}$$

in (15) and simplifying we get

$$X^2 = ac_0T^2 + i\alpha\beta - (k + 4)\beta^2$$

which is satisfied by  $T = 1, X = \alpha + i(k + 2)\beta$

In view of (16) and (13), it is seen that

$$c_1 = 4\alpha + i\beta(4k + 7)$$

Note that  $(a, c_0, c_1)$  represents diophantine 3-tuple with property  $D(i\alpha\beta - (k + 4)\beta^2)$

Taking  $(a, c_1)$  and employing the above procedure, it is seen that the triple  $(a, c_1, c_2)$  where

$$c_2 = 9\alpha + i\beta(9k + 20)$$

exhibits diophantine 3-tuple with property  $D(i\alpha\beta - (k + 4)\beta^2)$

Taking  $(a, c_2)$  and employing the above procedure, it is seen that the triple  $(a, c_2, c_3)$  where

$$c_3 = 16\alpha + i\beta(16k + 39)$$

exhibits diophantine 3-tuple with property  $D(i\alpha\beta - (k + 4)\beta^2)$

Taking  $(a, c_3)$  and employing the above procedure, it is seen that the triple  $(a, c_3, c_4)$  where

$$c_4 = 25\alpha + i\beta(25k + 64)$$

exhibits diophantine 3-tuple with property  $D(i\alpha\beta - (k + 4)\beta^2)$

The repetition of the above process leads to the generation of sequence of diophantine 3-tuples whose general form is given by  $(a, c_{\partial-1}, c_{\partial})$  where

$$c_{\partial-1} = \partial^2\alpha + i\beta((k+3)\partial^2 - 2\partial - 1), \partial = 1, 2, 3, \dots$$

**Table 4: Numerical Examples with property  $D(i\alpha\beta - (k+4)\beta^2)$**

<b>k</b>	<b><math>\alpha</math></b>	<b><math>\beta</math></b>	<b><math>(a, c_0, c_1)</math></b>	<b><math>(a, c_1, c_2)</math></b>	<b><math>(a, c_2, c_3)</math></b>
1	3	2	$(3+i8, 3+i2, 12+i22)$	$(3+i8, 12+i22, 27+i58)$	$(3+i8, 27+i58, 48+i110)$
2	4	3	$(4+i15, 4+i6, 16+i45)$	$(4+i15, 16+i45, 36+i114)$	$(4+i15, 36+i114, 64+i213)$
3	5	4	$(5+i24, 5+i12, 20+i76)$	$(5+i24, 20+i76, 45+i188)$	$(5+i24, 45+i188, 80+i348)$
4	6	5	$(6+i35, 6+i20, 24+i115)$	$(6+i35, 24+i115, 54+i280)$	$(6+i35, 54+i280, 96+i515)$
5	7	6	$(7+i48, 7+i30, 28+i162)$	$(7+i48, 28+i162, 63+i390)$	$(7+i48, 63+i390, 112+i714)$

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