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Observations on the Pell Equation $x^2 = 3(y^2+y)+1$

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ABSTRACT

This paper concerns with the problem of obtaining non-zero distinct integer solutions to the positive pell equation represented by the binary quadratic equation $x^2 = 3(y^2+y)+1$. A few interesting relations among the solutions are presented. Further, by considering suitable linear combinations among the solutions of the considered hyperbola, the other choices of hyperbolas, parabolas, 2^{nd} order Ramanujan numbers, sequence of diophantine 3-tuples with suitable property are presented.A general formula for generating sequence of integer solutions based on the given solution is illustrated.

Keywords— *Positive pell equation, binary quadratic, hyperbola, parabola,2^{nd} order Ramanujan numbers, sequence of diophantine 3-tuples*

1. INTRODUCTION

One of the areas of Number theory that has attracted many mathematicians since antiquity is the subject of diophantine equations. A diophantine equation is a polynomial equation in two or more unknowns such that only the integer solutions are determined. No doubt that diophantine equation possess supreme beauty and it is the most powerful creation of the human spirit. A Pell equation is a type of non-linear diophantine equation in the form $y^2 - Dx^2 = \pm 1$ where $D > 0$ and square-free. The above equation is also called the Pell-Fermat equation. In Cartesian co-ordinates, this equation has the form of a hyperbola. The binary quadratic diophantine equation having the form

$$y^2 = Dx^2 + N \quad (N > 0\ ;\ D > 0, \text{ a non-square integer})$$

is referred to as the positive form of the Pell equation and one may refer [1-10] for a few illustrations on Pell equations. As quadratic Diophantine equations are rich in variety, the above references motivated us for determining integer solutions to other choices of positive Pell equation.

In this communication, the Pell equation $x^2 = 3(y^2 + y) + 1$ is considered for obtaining non-zero distinct integer solutions. Further, by considering suitable linear combinations among the solutions of the considered hyperbola, the other choices of hyperbolas, parabolas,2^{nd}order Ramanujan numbers, sequence of diophantine 3-tuples with suitable property are presented. A general formula for generating sequence of integer solutions based on the given solution is illustrated.

2. METHOD OF ANALYSIS

The hyperbola represented by the non-homogeneous quadratic equation under consideration is

$$x^2 = 3(y^2 + y) + 1 \tag{1}$$

Treating (1) as a quadratic in *y* and solving for *y*,
we get

$$y = \frac{-3 \pm \sqrt{12x^2 - 3}}{6} \tag{2}$$

Let

$$Y^2 = 12x^2 - 3 \tag{3}$$

The smallest positive integer solution to (3) is $x_0=1$, $Y_0=3$

To find the other solutions to (1), consider the corresponding Pellian equation given by

$$Y^2=12x^2+1 \tag{4}$$

Where the general solution $\tilde{x}_n, \tilde{Y}_n$ is

$$\tilde{Y}_n=\frac{1}{2}f_n$$

$$\tilde{x}_n=\frac{1}{4\sqrt{3}}g_n$$

where

$$f_n=\left(7+4\sqrt{3}\right)^{n+1}+\left(7-4\sqrt{3}\right)^{n+1}$$

$$g_n=\left(7+4\sqrt{3}\right)^{n+1}-\left(7-4\sqrt{3}\right)^{n+1} \quad , \quad n=0,1,2,............$$

Employing the lemma of Brahmagupta between the solutions $\left(x_0, Y_0\right)$ & $\left(\tilde{x}_n, \tilde{Y}_n\right)$,the general solution $\left(x_{n+1}, Y_{n+1}\right)$ to (3) is given by

$$x_{n+1}=x_0\tilde{Y}_n+Y_0\tilde{x}_n$$

$$=\frac{1}{2}f_n+\frac{\sqrt{3}}{4}g_n \tag{5}$$

$$Y_{n+1}=Y_0\tilde{Y}_n+Dx_0\tilde{x}_n$$

$$=3*\frac{1}{2}f_n+\sqrt{3}*g_n$$

In view of (2) and taking the positive sign before the square-root on the R.H.S. of (2),
we have

$$y_{n+1}=\frac{1}{12}\left(3f_n+2\sqrt{3}g_n-6\right) \tag{6}$$

Thus, (5) and (6) represented the integer solutions to (1).
A few numerical solutions to (1) are presented in Table below:

Table: Numerical solutions

N	x_{n+1}	y_{n+1}
-1	1	0
0	13	7
1	181	104
2	2521	1455
3	35113	20272

3. OBSERVATIONS

The x-values are odd primes whereas y-values are alternatively odd and even.
A few interesting relations among the solutions are given below:

- $x_{n+3}-14x_{n+2}+x_{n+1}=0$
- $y_{n+3}-14y_{n+2}+y_{n+1}=6$
- $12y_{n+1}+6=x_{n+2}-7x_{n+1}$
- $12y_{n+2}+6=-x_{n+1}+7x_{n+2}$
- $12y_{n+3}+6=-7x_{n+1}+97x_{n+2}$
- $168y_{n+1}+84=x_{n+3}-97x_{n+1}$
- $24y_{n+2}+12=x_{n+3}-x_{n+1}$
- $168y_{n+3}+84=97x_{n+3}-x_{n+1}$
- $4x_{n+1}=y_{n+2}-7y_{n+1}-3$
- $4x_{n+2}=7y_{n+2}-y_{n+1}+3$
- $4x_{n+3}=97y_{n+2}-7y_{n+1}+45$
- $56x_{n+1}=y_{n+3}-97y_{n+1}-48$

- $8x_{n+2} = y_{n+3} - y_{n+1}$
- $56x_{n+3} = 97y_{n+3} - y_{n+1} + 48$
- $x_{n+1} = 7x_{n+2} - 12y_{n+2} - 6$
- $x_{n+3} = 7x_{n+2} + 12y_{n+2} + 6$
- $y_{n+1} = 7y_{n+2} - 4x_{n+2} + 3$
- $y_{n+3} = 7y_{n+2} + 4x_{n+2} + 3$

Expressions representing square integers:

- $[15x_{2n+2} - x_{2n+3} + 2]$
- $\frac{1}{14}[209x_{2n+2} - x_{2n+4} + 28]$
- $[2y_{2n+3} - 26y_{2n+2} - 10]$
- $\frac{1}{7}[y_{2n+4} - 181y_{2n+2} - 76]$

Expressions representing cubical integers:

- $[15x_{3n+3} - x_{3n+4} + 3(15x_{n+1} - x_{n+2})]$
- $\frac{1}{14}[209x_{3n+3} - x_{3n+5} + 3(209x_{n+1} - x_{n+3})]$
- $[2y_{3n+4} - 26y_{3n+3} + 6y_{n+2} - 78y_{n+1} - 48]$
- $\frac{1}{7}[y_{3n+5} - 181y_{3n+3} + 3y_{n+3} - 543y_{n+1} - 360]$

Expressions representing biquadratic integers:

- $(15x_{4n+4} - x_{4n+5}) + 4(15x_{n+1} - x_{n+2})^2 - 2$
- $(15x_{4n+4} - x_{4n+5}) + 4(15x_{2n+2} - x_{2n+3} + 2) - 2$
- $\frac{1}{14}(209x_{4n+4} - x_{4n+6}) + \frac{1}{49}(209x_{n+1} - x_{n+3})^2 - 2$
- $\frac{1}{14}(209x_{4n+4} - x_{4n+6}) + \frac{2}{7}(209x_{2n+2} - x_{2n+4} + 28) - 2$
- $(2y_{4n+5} - 26y_{4n+4} - 14) + 16(y_{n+2} - 13y_{n+1} - 6)^2 - 2$
- $(2y_{4n+5} - 26y_{4n+4} - 14) + 8(y_{2n+3} - 13y_{2n+2} - 5) - 2$
- $\frac{1}{7}(y_{4n+6} - 181y_{4n+4} - 90) + \frac{4}{49}(y_{n+3} - 181y_{n+1} - 90)^2 - 2$
- $\frac{1}{7}(y_{4n+6} - 181y_{4n+4} - 90) + \frac{4}{7}(y_{2n+4} - 181y_{2n+2} - 76) - 2$

Employing linear combinations among the solutions, one obtains solutions to other choices of hyperbolas

Choice1: Let $Y = x_{n+2} - 13x_{n+1}, X = 15x_{n+1} - x_{n+2}$

Note that (X, Y) satisfies the hyperbola

$$3X^2 - 4Y^2 = 12$$

Choice2: Let $Y = x_{n+3} - 181x_{n+1}, X = 209x_{n+1} - x_{n+3}$

Note that (X, Y) satisfies the hyperbola

$$3X^2 - 4Y^2 = 48*49$$

Choice3: Let $Y = 15y_{n+1} - y_{n+2} + 7, X = 2y_{n+2} - 26y_{n+1} - 12$

Note that (X, Y) satisfies the hyperbola

$$4X^2 - 3Y^2 = 4$$

Choice4: Let $Y = 209y_{n+1} - y_{n+3} + 104, X = y_{n+3} - 181y_{n+1} - 90$

Note that (X, Y) satisfies the hyperbola

$$4X^2 - 3Y^2 = 49*16$$

Employing linear combinations among the solutions, one obtains solutions to other choices of parabolas

Choice1: Let $Y = x_{n+2} - 13x_{n+1}, X_1 = 15x_{2n+2} - x_{2n+3} + 2$

Note that (Y, X_1) satisfies the parabola

$$3X_1 - 4Y^2 = 12$$

Choice2: Let $Y = x_{n+3} - 181x_{n+1}, X_1 = 209x_{2n+2} - x_{2n+4} + 28$

Note that (Y, X_1) satisfies the parabola

$$21X_1 - 2Y^2 = 21*56$$

Choice3: Let $Y = 209y_{n+1} - y_{n+3} + 104, X_1 = y_{2n+4} - 181y_{2n+2} - 76$

Note that (Y, X_1) satisfies the parabola

$$28X_1 - 3Y^2 = 4*196$$

Choice4: Let $Y = 15y_{n+1} - y_{n+2} + 7, X_1 = y_{2n+3} - 13y_{2n+2} - 5$

Note that (Y, X_1) satisfies the parabola

$$2X_1 - 3Y^2 = 4$$

Considering suitable values of x_{n+1} and y_{n+1} ,one generates 2^{nd} order Ramanujan numbers with base integers as real integers

For illustration, consider

$$y_2 = 104 = 1\times 104 = 2\times 52 = 4\times 26 = 8x13 \qquad (*)$$

Now, $1\times 104 = 2\times 52$

$\rightarrow (104+1)^2 + (52-2)^2 = (104-1)^2 + (52+2)^2$

$\rightarrow 105^2 + (50)^2 = (103)^2 + 54^2 = 13525$

$1\times 104 = 4\times 26$

$\rightarrow (104+1)^2 + (26-4)^2 = (104-1)^2 + (26+4)^2 = 11509$

$1\times 104 = 8\times 13$

$\rightarrow (104+1)^2 + (13-8)^2 = (104-1)^2 + (13+8)^2 = 11050$

$2\times 52 = 4\times 26$

$\rightarrow (52+2)^2 + (26-4)^2 = (52-2)^2 + (26+4)^2 = 3400$

$2\times 52 = 8\times 13$

$\rightarrow (52+2)^2 + (13-8)^2 = (52-2)^2 + (13+8)^2 = 2941$

$4\times 26 = 8\times 13$

$\rightarrow (26+4)^2 + (13-8)^2 = (26-4)^2 + (13+8)^2 = 925$

Also,

$2\times 52 = 4\times 26 \rightarrow 27^2 - 25^2 = 15^2 - 11^2$

$\rightarrow 27^2 + 11^2 = 15^2 + 25^2 = 850$

Thus, 13525 ,11509,11050,3400,2941,925,850 represent 2^{nd} order Ramanujan numbers with base integers as real integers.

Considering suitable values of x_{n+1} & y_{n+1} , one generates 2^{nd} order Ramanujan numbers with base integers as Guassian integers. For illustration, consider again y_2 represented by (*)

Now, $1\times 104 = 2\times 52 \rightarrow (1+i104)^2 + (2-i52)^2 = (1-i104)^2 + (2+i52)^2 = -13520$

Also, $1\times 104 = 2\times 52 \rightarrow (104+i)^2 + (52-i2)^2 = (104-i)^2 + (52+i2)^2 = 13520$

Note that -13520 & 13520 represent 2^{nd} order Ramanujan numbers with base integers as Gaussian integers.

In a similar manner, other 2^{nd} order Ramanujan numbers are obtained

Formation of sequence of Diophantine 3-tuples:

Consider the solution to (1) given by

$$x_1 = 13 = a\,(say),\, y_1 = 7 = c_0\,(say)$$

It is observed that

$ac_0 + k^2 - 91 = k^2$, a perfect square

The pair (a, c_0) represents diophantine 2-tuple with property D$(k^2 - 91)$.

If c_1 is the 3^{rd} tuple, then it satisfies the system of double equations

$$13c_1 + k^2 - 91 = p^2 \quad (1^*)$$

$$7c_1 + k^2 - 91 = q^2 \quad (2^*)$$

Eliminating c_1 between (1*) and (2*), we have

$$6(k^2 - 91) = 13q^2 - 7p^2 \quad (3^*)$$

Taking

$$p = X + 13T,\, q = X + 7T \quad (4^*)$$

in (3*) and simplifying, we get

$$X^2 = 91T^2 + k^2 - 91$$

which is satisfied by

$$X = k, T = 1$$

In view of (4*) and (1*), it is seen that

$$c_1 = 2k + 20$$

Note that $(13, 7, 2k + 20)$ represents diophantine 3-tuple with property $D(k^2 - 91)$.

The process of obtaining sequences of diophantine 3-tuples with property $D(k^2 - 91)$ is illustrated below:

Let M be a 3*3 square matrix given by

$$M = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix}$$

Now

$$(13, 7, 2k + 20)M = (13, 2k + 20, 4k + 59)$$

Note that

$$13*(2k + 20) + (k^2 - 91) = perfect\ square$$

$$13*(4k + 59) + (k^2 - 91) = perfect\ square$$

$$(2k + 20)*(4k + 59) + (k^2 - 91) = perfect\ square$$

Therefore, the triple $(13, 2k + 20, 4k + 59)$ represents diophantine 3-tuple with property $D(k^2 - 91)$.The repetition of the above process leads to sequences of diophantine 3-tuples whose general form (a, c_{S-1}, c_S) is given by

$$\left(13, 13s^2 + (2k - 26)s - 2k + 20, 13s^2 + 2ks + 7\right), \text{ s=1,2,3……..}$$

A few numerical illustrations are given in Table below:

Table: Numerical illustrations

K	(a, c_0, c_1)	(a, c_1, c_2)	(a, c_2, c_3)	$D(k^2 - 91)$
0	(13,7,20)	(13,20,59)	(13,59,124)	D(-91)
1	(13,7,22)	(13,22,63)	(13,63,130)	D(-90)
2	(13,7,24)	(13,24,67)	(16,114,136)	D(-87)

It is noted that the triple $(c_{s-1}, c_s + 13, c_{s+1})$, s=1,2,3……

forms a arithmetic progression.

In a similar way, one may generate sequences of diophantine 3-tuples with suitable property through the other solutions to (1).

Generation of solutions:

Let (x_0, y_0) represents any given solution to (1).

Consider the second solution (x_1, y_1) to (1) given by

$$x_1 = 2h - x_0,\ y_1 = y_0 + h \tag{7}$$

Substituting (7) in (1) and simplifying, one obtains

$$h = 4x_0 + 6y_0 + 3$$

In view of (7), we have

$$x_1 = 7x_0 + 12y_0 + 6,\ y_1 = 4x_0 + 7y_0 + 3$$

which is written in the form of matrix as

$$(x_1, y_{1,} 1)^t = \begin{pmatrix} 7 & 12 & 6 \\ 4 & 7 & 3 \\ 0 & 1 & 1 \end{pmatrix} (x_0, y_0, 1)^t$$

where t is the transpose. The repetition of the above process leads to the general solution to (1) as

$$(x_{n+1}, y_{n+1}, 1)^t = \begin{pmatrix} Y_n & 3X_n & \frac{3X_n}{2} \\ X_n & Y_n & \frac{Y_n - 1}{2} \\ 0 & 0 & 1 \end{pmatrix} (x_0, y_0, 1)^t, n = 0,1,2,...$$

where

$$Y_n = \frac{1}{2}((7 + 4\sqrt{3})^{n+1} + (7 - 4\sqrt{3})^{n+1})$$

$$X_n = \frac{1}{2\sqrt{3}}((7 + 4\sqrt{3})^{n+1} - (7 - 4\sqrt{3})^{n+1})$$

4. CONCLUSION

As quadratic Diophantine equations are rich in variety, the readers may attempt for finding integer solutions to quadratic Diophantine equations with two or more variables along with suitable properties.

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