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Radial radio number of uniform cyclic and wheel split graphs

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ABSTRACT

A radial radio labelling h, of a connected graph $G = (V, E)$ is an assignment of non-negative integers to the vertices of G satisfying the radial radio condition $d(u, v) + |h(u)–h(v)| \geq 1 + rad(G)$, for any two distinct vertices $u, v \in V(G)$, where $rad(G)$ denote the radius of the graph G. The span of a radial radio labeling h is the largest integer in the range of h and is denoted by $rr(h)$. The radial radio number of G, denoted by $rr(G)$, is the minimum span taken over all radial radio labelings of G. In this paper, we have obtained the radial radio number of certain wheel related graphs such as the graph KDW(r), $HW(r)$, $SW(r)$, uniform n −wheel split graph and uniform r −cyclic split graphs.

Keywords: *Labelling, Radial Radio labelling, Radial radio number, Uniform n-wheel spilt graphs, uniform r-cyclic split graphs.*

1. INTRODUCTION

In the early 1980's, Hale [7] introduced a Graph theory model for radio frequency assignment problems. In 2001 Chartrand et al. [1] were motivated by regulations for channel assignments of FM radio stations to introduce radio labelling for connected graphs. A *radio labelling* of a connected graph G is an injection h from the vertices of G to the natural numbers such that $d(u, w) + |h(u) - h(w)| \geq 1 + diam(G)$ for every pair of vertices u and w of G. The *radio number of h* denoted by $rn(h)$, is the maximum number assigned to any vertex of G. The *radio number of G*, denoted $rn(G)$, is the minimum value of $rn(h)$, taken over all labellings h of G. In 2014, Ponraj et.al [5] introduced a variation of radio labelling called radio mean labelling. A *radio mean labelling* of a connected graph G is an injection h from the vertices of G to the natural numbers such that $d(u, w) + \left|\left\lceil\frac{h(u)+h(v)}{2}\right\rceil\right| \geq 1 + diam(G)$ for every pair of vertices u and w of G. The *radio mean number of h* denoted by $rn(h)$, is the maximum number assigned to any vertex of G. The *radio mean number of G*, denoted $rn(G)$, is the minimum value of $rn(h)$, taken over all labellings h of G. In 2017, Hemalatha et.al [3] introduced another labelling called radio geometric mean labelling, by replacing the mean condition $\left|\left\lceil\frac{h(u)+h(v)}{2}\right\rceil\right|$ by geometric mean condition $\left|\left\lceil\sqrt{h(u)h(v)}\right\rceil\right|$. Recently in 2019, Avadayappan et.al [6] replace the diameter by radius and introduce a new labelling called radial radio labelling. The formal graph theoretical definition is as follows:

A radial radio labelling h, of a connected graph $G = (V, E)$ is an assignment of non-negative integers to the vertices satisfying the radial radio condition $d(u, v) + |h(u)–h(v)|1 + rad(G)$, for any two distinct vertices $u, v \in V(G)$, where $rad(G)$ denote the radius of the graph G. The span of a radial radio labeling h is the largest integer in the range of f and is denoted by $rr(h)$ The radial radio number of G, denoted by $rr(G)$, is the minimum span taken over all radial radio labelling of G. Avadayappan et.al [6] proved that, for any simple connected graph G that $rr(G) \geq \omega(G)$. For any graph G with $m \geq 1$, there is a graph G with $\omega = 3$ and $rr(G) = m + \omega$. Also, for any graph G with $\omega \geq 4$, there exists a graph G with $rr(G) = \omega + 1$.

In this paper, we have obtained the bounds for the radial radio number of certain Uniform cyclic and wheel split graphs.

1.1 Preliminaries

In this section, we have listed few relevant results and definitions which are used in this paper.

***Definition 1*:** Let G be a connected graph and let v be a vertex of G. The *eccentricity* $e(v)$of v is the farthest vertex from v. Thus $e(v) = \max\{d(u, v) \forall u \in V(G)$.

***Definition* 2:** The *diameter of G* is the maximum eccentricity of the vertices of G. It is denoted by $diam(G)$.

***Definition* 3:** The *radius of G* is the minimum eccentricity of the vertices of G. It is denoted by $rad(G)$.

***Definition* 4:** The center of graph G is defined as the set of vertices having eccentricity equal to the radius of the graph G.

***Remark* 1:** For any graph G, $rad(G) \leq e(v) \leq diam(G)$ for all vertices $v \in V(G)$.

Bharathi et.al [2] and Kins Yenoke [4] were introduced the following wheel related graphs.

***Definition* 5: [4]** Let $x_i, 1 \leq i \leq n$ be the vertices of the Complete graph K_n. Let x_i be adjacent to $w_i, 1 \leq i \leq n$. Subdivide each edge $x_i w_i$ by $u_{i,} 1 \leq i \leq n$. Let W_i be a wheel with hub w_i. The graph obtained is denoted by $KDW(r), n > 6$.

***Remark* 3:** The number of vertices in $KDW(r)$ is $n(r+3)$ and the number of edges is $2n(r+1) + \frac{n(n+1)}{2}$. Also the radius and diameter of $KDW(r)$ are 4 and 7 respectively. The center of the graph is K_n.

***Definition* 6: [4]** Let H_{n+1} be the helm with the pendant vertices $u_i, 1 \leq i \leq n$. Let $W_{r+1}^i = C_r^i + K_1$ be wheels with hubs $u_i, 1 \leq i \leq n$ respectively. The graph constructed is denoted by $HW(r), n > 11$.

***Remark* 4:** The number of vertices and edges of $HW(r)$ are $n(r+2)+1$ and $2n(r+1)+n$ respectively. Its diameter is 6 and radius is 4. The center vertex of the inner wheel is the only center of the graph.

***Definition* 7: [4]** Let $w_i, 1 \leq i \leq n$ be the vertices of a star graph $S_{n+1}$with hub at x. Let u_i be adjacent to $w_i, 1 \leq i \leq n$ respectively. The graph obtained is denoted by $SW(r)$. The number of vertices in $SW(r)$ is $n(r+2)+1$ and the number of edges is $2n(r+1)$.

***Remark* 5:** The diameter of $SW(r)$ is 6. Also, the center of the graph is a single vertex of radius 3, which is the center of the inner star graph S_{n+1}.

***Definition* 8: [2]** A *uniform* r *−cyclic split graph* $KC(r)$ is a graph in which the deletion of nmr edges partitions the graph into a complete K_n and nm independent cycles of length r. If each of the nm cycles of length r is shrunk to a point, then the uniform r −cyclic split graph reduces to the standard split graph. There are k wheels attached to each vertex of the complete graph. The number of vertices and edges are $n(mr+1)$ and $2nmr + \frac{n(n-1)}{2}$ respectively.

***Remark* 6:** The diameter and radius of KC(r) are 3 and 2 respectively.

***Definition* 9:** A *uniform n-wheel split graph KW(r)* is a graph in which the deletion of n edges partitions the graph into a complete graph and n independent wheels $W_{r+1}, n > 4$. This graph can be thought of as a generalization of the standard split graph in the sense that the elements of the independent set are replaced by wheels here.

***Remark* 7:** The number of vertices in KW(r) is $n(r+2)$ and the number of edges is $n\left(2r + \frac{n-1}{2} + 1\right)$. Further, the radius and diameter are 3 and 5 respectively.

1.2 Radial Radio Labelling of $KDW(r)$, $HW(r)$, $SW(r)$ uniform n −wheel split graph and uniform r −cyclic split graphs
In this section we have determined the radial radio labelling of certain special wheel related graphs with radius 2, 3 and 4.

***Theorem* 3.1:** The radial radio number of the graph $KDW(r)$ with radius 4, satisfies $rr(KDW(r)) \leq 3(r+2n), n > 4$.

***Proof*:** Define a mapping h from the vertex set of $KDW(r)$ to the set of non-negative integers as follows:

$$h(v_{r(j-1)+2i-1}) = 3(i-1), i = 1,2..\left\lceil\frac{r}{2}\right\rceil, j = 1,2..n,$$

$$h(v_{r(j-1)+2i}) = 3\left(\left\lceil\frac{r}{2}\right\rceil - 1\right) + 3i, i = 1,2..\left\lfloor\frac{r}{2}\right\rfloor, j = 1,2..n,$$

$$h(x_i) = 3(r-1) + 4(i-1) + 2, h(w_i) = 3r + 4n - 2,\ h(u_i) = 3r + 4\,n + 2i\ ,\ i = 1,2 \dots n.$$ See Fig.1.

Next, we claim that h satisfies the radial radio labelling condition.

That is to prove that $d(u,w) + |\,h\,(u) - h(w)| \geq 1 + rad(KDW(r)) = 5\ \forall u, w \in V(KDW(r))$.

Let u and w be any two vertices in the graph $KDW(r)$.

Case 1: Suppose u and w lies on the wheel graphs.

Case 1.1: If $u = v_{r(k-1)+2s-1}$ and $w = v_{r(l-1)+2m-1}$, $1 \leq k, l \leq n$, $1 \leq s \neq m \leq \left\lceil\frac{r}{2}\right\rceil$, then the distance between them is at least 2. Also $h(u) = 3(s-1)$ and $h(w) = 3(m-1)$. Hence, the radial radio labelling condition becomes $d(u,w) + |\,h\,(u) - h(w)| \geq 2 + |3(s-1) - (3(m-1))| \geq 2 + 3 = 5$, since $s \neq m$.

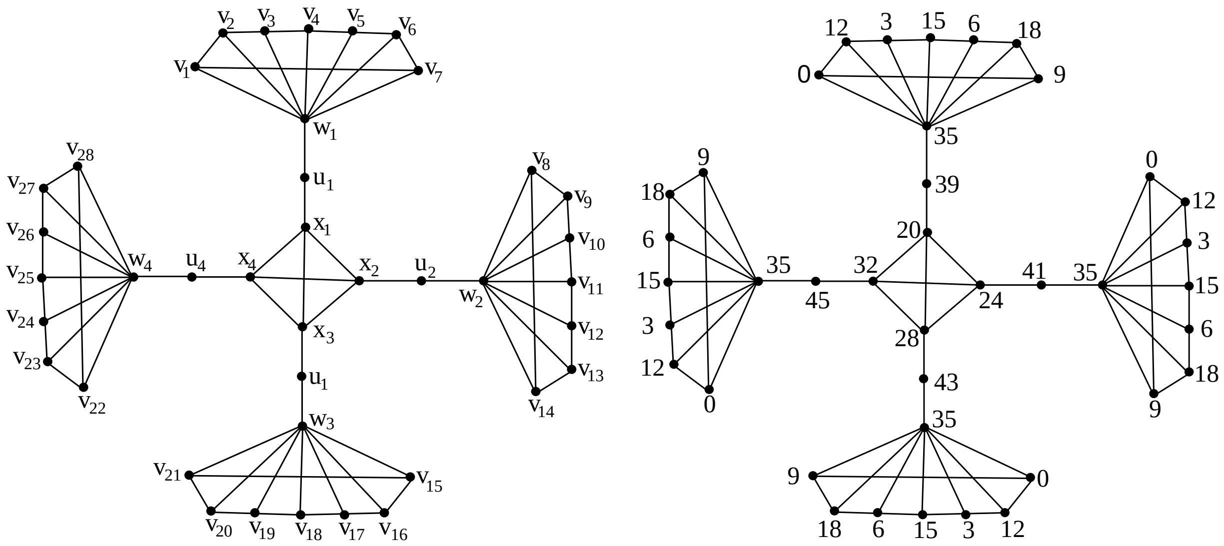


Fig. 1: A radial radio labelling of a graph KDW(r) with r = 7 and n = 4 which attains the bound.

Case 1.2: If $u = v_{r(k-1)+2s}$ and $w = v_{r(l-1)+2m}$, $1 \le k, l \le n$, $1 \le s \ne m \le \left\lceil \frac{r}{2} \right\rceil$, then $d(u,w) \ge 2$. Also the modulus difference between $h(u)$ and $h(w)$ is at least $3(s-m)$ .Therefore, $d(u,w) + |\, h\,(u) - h(w)| \ge 2 + 3 = 5,$ since $s \ne m$.

Case 1.3: Let $u = v_{r(k-1)+2s-1}$ and $w = v_{r(l-1)+2m}$, then $h(u) = 3(s-1), 1 \le s \le \left\lceil \frac{r}{2} \right\rceil$ and $h(w) = 3\left(\left\lceil \frac{r}{2} \right\rceil - 1\right) + 3m,\ 1 \le m \le \left\lceil \frac{n}{2} \right\rceil$. Again, the distance between u and w is either 1, 2 or 7.

When $d(u,w) = 1,$ then the moduls difference between $h(u)$ and $h(v)$ *is* at least 6, since $n > 4$. Therefore, $d(u,w) + |\, h\,(u) - h(w)| \ge 5$.

Also, if $d(u,w) = 2$ *or* 7, then $|h(u) - h(w)| \ge 3$. Therefore, $d(u,w) + |\, h\,(u) - h(w)| \ge 2 + 3 = 5$.

Case 2: Suppose u and w are any two vertices in the complete graph, then $u = x_k$ and $w = x_l$, $1 \le k \ne l \le n$. Also $d(u,w) = 1$ and $h(u) = 3(r-1) + 4(k-1) + 2, h(u) = 3(r-1) + 4(l-1) + 2$. Therefore, $d(u,w) + |\, h\,(u) - h(w)| \ge 1 + |\,4(k-l)| \ge 5,$ since $k \ne l$.

Case 3: If u and w are any two vertices in the subdivision of $x_i w_i$, $1 \le i \le n$, then $u = u_k$ and $w = u_l$, $1 \le k \ne l \le n$.

Here, $h(u) = 3r + 4\,n + 2k$, $h(w) = 3r + 4\,n + 2l$ and the distance between them is exactly 3. Hence the radial radio labelling condition becomes $d(u,w) + |\, h\,(u) - h(w)| = 3 + |\,2(k-l)| \ge 5,$ since $k \ne l$.

Case 4: Suppose u and w are any two hub vertices of the wheels, then h maps to the same number $3r + 4n - 2$. But the distance between them is always 5. Hence h satisfies the radial radio labelling condition in this case.

Case 5: Let u and w be vertices of the wheel graph and complete graph respectively. Hence, the function values of u and w are $h(u) = 3(k-1)$ and $h(x_m) = 3(r-1) + 4(m-1) + 2$ respectively, where $1 \le k \le r$, $1 \le m \le n$. Also, the distance between the two vertices is equal to 3 or 4. Therefore, $|\, h\,(u) - h(w)| \ge 2$ and thus the radial radio condition is satisfied.

Case 6: Assuming u is a vertex in any wheel and $w = w_l$, $1 \le l \le n$, then the difference in function value is given by $|h(u) - h(w)| = |3(k-1) - (3r + 4n - 2)|$. Also, since $d(u,w) \ge 1$ and k lies between 1 and r, we get $d(u,w) + |\, h\,(u) - h(w)| \ge 1 + 4n - 5 > 5,$ since $n > 4$.

Case 7: If $u = v_k, 1 \le k \le nr$ and $w = u_l$, $1 \le l \le n$, then $d(u,w) \ge 2$ and h maps u, w to $3(k-1)$, $3r + 4\,n + 2l$ respectively. Hence $d(u,w) + |\, h\,(u) - h(w)| \ge 2 + 4n + 2l - 3 > 5$.

Case 8: Suppose u is any hub vertex and w is any complete graph vertex, then the distance between them is at least 1. Also, $h(u) = 3r + 4n - 2$ and $h(u_i) = 3r + 4\,n + 2m$. Therefore $d(u,w) + |\, h\,(u) - h(w)| \ge 1 + 4 = 5$.

Case 10: Suppose $u = x_l$ is a vertex in the complete graph and $w = u_m$, $1 \le l, m \le n$, then $d(u,w) \ge 2$ and $|\, h\,(u) - h(w)| \ge 7$. $d(u,w) + |\, h\,(u) - h(w)| \ge 9 > 5$.

Thus, h is a valid radial radio labelling.

Also, the vertex u_n attains the maximum value $h(u_n) = 3r + 4\,n + 2n = 3(r + 2n)$. Thereby we are proving the theorem by attaining the result $rr\big(KDW(r)\big) \le 3(r + 2n), n > 4$.

***Theorem* 3.2:** Let G be a graph HW(r). Then the radial radio number of G satisfies $rr(G) \leq 2(r+n)+2, n>3$.

***Proof*:** First we name the vertices of $C_r^i, i=1,2..n$ in the graph HW(r) as $v_1, v_2 \dots v_{nr}$. Next, we name the vertices of the center wheel as $w_1, w_2 \dots w_{n+1}$. The remaining vertices are named as in definition 2.6.

Define a mapping $h: V(\mathrm{HW(r)}) \rightarrow N \cup \{0\}$ as follows:

$$h(v_{r(j-1)+2i-1}) = 2i-1, \qquad i=1,2 \dots \left\lceil \frac{r}{2} \right\rceil, j=1,2..n,$$

$$h(v_{r(j-1)+2i}) = 2\left\lceil \frac{r}{2} \right\rceil - 1 + 2i, i=1,2 \dots \left\lfloor \frac{r}{2} \right\rfloor, j=1,2..n,$$

$$h(w_{2i-1}) = 2(r+i)-1, i=1,2 \dots \left\lceil \frac{n}{2} \right\rceil, h(w_{2i}) = 2\left(r+\left\lceil \frac{n}{2} \right\rceil\right) - 1 + 2i, i=1,2 \dots \left\lfloor \frac{n}{2} \right\rfloor, h(w_{n+1}) = 0$$

$$h(u_i) = 2(r+n)+1, i=1,2 \dots \left\lceil \frac{n}{2} \right\rceil, h(u_i) = 2(r+n)+2, i=1,2 \dots \left\lfloor \frac{n}{2} \right\rfloor. \text{ See Fig. 2 (a).}$$

***Theorem* 3.3:** For any $n>1$, the radial radio numer of $SW(r)$is given by $rr(SW(r)) \leq 2r+3n$.

***Proof*:** we name the vertices of $SW(r)$ as same as the vertices of the graph HW(r).

Define a mapping $h: V(SW(r)) \rightarrow N \cup \{0\}$ as follows:

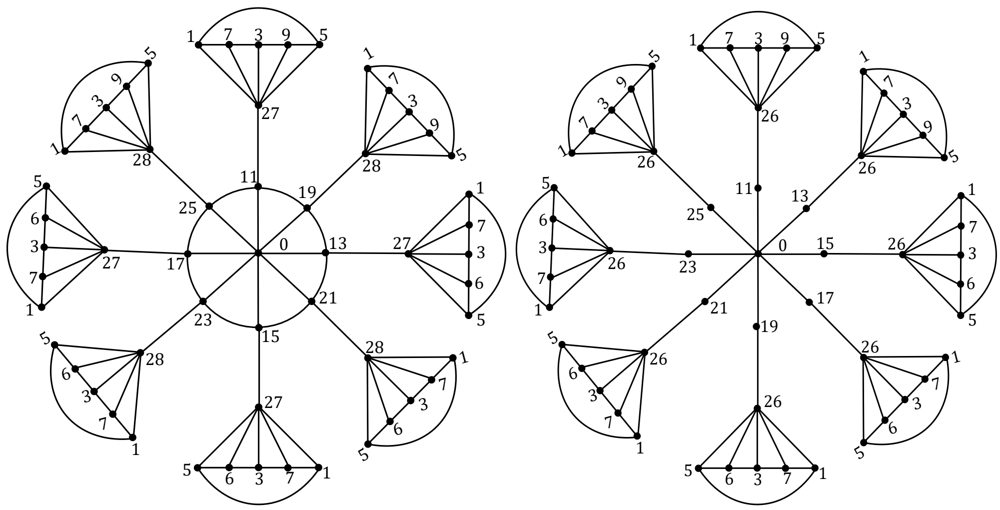
$$h(v_{r(j-1)+2i-1}) = 2i-1, \qquad i=1,2 \dots \left\lceil \frac{r}{2} \right\rceil, j=1,2..n,$$

$$h(v_{r(j-1)+2i}) = 2\left\lceil \frac{r}{2} \right\rceil - 1 + 2i, i=1,2 \dots \left\lfloor \frac{r}{2} \right\rfloor, j=1,2..n,$$

$$h(w_i) = 2(r+i)-1, i=1,2 \dots n, \; h(w_{n+1}) = 0,$$

$$h(u_i) = 2(r+n), \; i=1,2 \dots n.$$

The above mapping and the Figure 2. (b) illustrates the theorem 3.3.



(a) (b)

Fig. 2: The graphs HW(5) and SW(r) with n = 8 and its radial radio labelling which attains the bound.

***Lemma* 3.1:** Let $G=(V,E)$ be a connected graph of order n with radius 3. Suppose there exists a vertex v in a clique in G of order k in G such that $deg\ (v) = \Delta(G) = s$, then the radial radio number of G is at least $s+l-1$.

***Proof*:** Let $V_1 = \{v_1, v_2 \dots v_l\}$ be the vertices of the clique in the graph G. Let $v \in V_1$ such that $deg\ (v) = \Delta(G) = s$. Since the radius of the graph is 3, for a radial radio labelling condition, we must assign $m+1$ distinct labels including the vertex v. Again, each of the l clique vertices must be labelled with a difference at least 2. Since already we have used l numbers in the adjacency of v for the clique vertices, we must label with l more distinct numbers. But if we assign radial radio labelling by starting and ending with clique vertex then we can reduce 2 more numbers. Hence the lower bound of the graph is $s+1+l-2 = s+l-1$.

***Theorem* 3.4:** Let KC(r) be a uniform r –cyclic split graph with $n(mr+1)$ vertices. Then,

$$rr(KC(r))) = mr + 2n - 2, m > 1.$$

Proof: First we name the vertices of the nm cycles and the complete graph as $\{v_1, v_2 \dots v_r, v_{r+1} \dots v_{mr}, v_{mr+1} \dots v_{nmr}\}$ and $\{w_1, w_2 \dots v_n\}$ respectively. Next, we define a mapping h from the vertex set of KC(r) to the non-negative integers as follows:

$$h(v_{mr(k-1)+(r(i-1)+j)}) = m(j-1) + i, i = 1,2 \dots m, j = 1,2 \dots r, k = 1,2 \dots n.$$

$$h(w_i) = mr + 2i, i = 1,2 \dots n-1, h(w_n) = 0.$$

Now we claim that $d(u,w) + |\,h\,(u) - h(w)| \geq 4,$ for all $u, w \in V(\text{KC(r)})$.

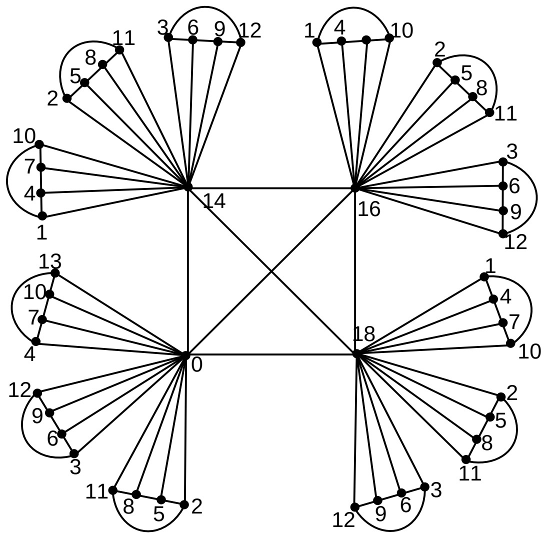


Fig. 3: A radial radio labelling of KC(4) with m = 3, n = 4 which attains the radial radio number

Case 1: Suppose u and w lies on the same cycle, then $u = v_{mr(k-1)+(r(i-1)+s)}$ and $w = v_{mr(k-1)+(r(i-1)+t)}$, $1 \leq k \leq n,\ 1 \leq i \leq m,\ 1 \leq s \neq t \leq r$. Therefore, $h(u) = m(s-1) + i,\ \ h(w) = m(t-1) + i$ and $d(u,w) \geq 1$. Hence the radial radio labelling condition becomes $d(u,w) +\ |\,h\,(u) - h(w)| \geq 1 + |m(s-t)| \geq 3$, since $m > 1$.

Case 2: Suppose u and w lies on the different cycle with the same hub vertex then $u = v_{mr(k-1)+(r(s-1)+j)}$ and $w = v_{mr(k-1)+(r(t-1)+j)}$, $1 \leq k \leq n,\ 1 \leq j \leq r, 11 \leq s \neq t \leq m$. Here, $h(u) = m(j-1) + s,\ h(w) = m(j-1) + t$ and $d(u,w) \geq 2$. Therefore $d(u,w) +\ |\,h\,(u) - h(w)| \geq 2 + |(s-t)| \geq 3$, since $s \neq t$.

Case 3: Suppose u and w lies on the different cycle with the same hub vertex then $u = v_{mr(s-1)+(r(i-1)+j)}$ and $w = v_{mr(t-1)+(r(i-1)+j)}$, $1 \leq i \leq m,\ 1 \leq j \leq r, 11 \leq s \neq t \leq n$. Therefore, the modulus difference of $h(u)$ and $h(v)$ is greater than or equal to zero. Also, the distance between them is exactly 3. Hence, $d(u,w) +\ |\,h\,(u) - h(w)| \geq 3$.

Case 4: Let u be a vertex on any cycle and w be a vertex in the complete graph.

Case 4.1: If $w \neq w_{n+1}$, then $h(u)$ takes the maximum value mr and $h(w)$ takes the minimum value $mr + 2$. But the difference between them is either 1 or 2. Hence $d(u,w) +\ |\,h\,(u) - h(w)| \geq 1 + |mr + 2 - mr| = 3$.

Case 4.2: If $w = w_{n+1}$, then $h(u)$ takes the minimum value 1 for the vertex v_1 which is at a distance 2 from the label 0. Hence the radial radio labelling condition holds in this case.

Case 5: Suppose u and w are any two vertices in the complete graph, then the distance between them is 1 and $|f(u) - f(v) \geq 2|$.

Hence $d(u,w) +\ |\,h\,(u) - h(w)| \geq 3$.

Therefore, the radial radio labelling of KC(r) satisfies**,** $rr(KC(r))) \leq mr + 2n - 2$ ________________________ (1)

Again, from Lemma 3.1, we have $\deg(w_{1)} = \Delta(KC(r)) = s = kr + n - 1$ and $l = n$.

Therefore, we get $rr(KC(r))) \geq s + l - 1 = mr + n - 1 + n - 1 = mr + 2n - 2$ ____________________________(2)

From equations (1) and (2) we get, $rr(KC(r))) = mr + 2n - 2$.

***Theorem* 5:** Let G be a uniform n-wheel split graph KW(r), then the radial radio number of G satisfies $\text{rr(KW(r))} \leq 2\text{r} + 4(\text{n} - 1), \text{n} > 1$.

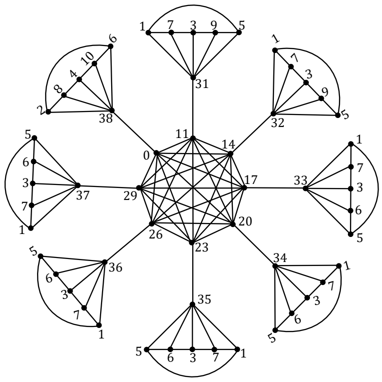


Fig. 4: A graph $\mathbf{KDW(5)}$ with $\mathbf{n = 8}$ and its radial radio labelling

Proof: The proof is left to the reader.

3. CONCLUSION

In this paper we have determined the radial radio labelling of certain uniform split and wheel graphs. Further the work is extended to the interconnection networks.

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