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Analytic continuation of Riemann zeta function

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ABSTRACT

In this paper, we present Riemann zeta function and its analytic continuation and functional equation. We will begin with the Gamma function and some properties of the Gamma function. Then we prove that the Riemann zeta function can be represented by Euler's product. We will present proof of analytic continuation of the zeta function using Gamma function. Also, we define a Jacobi Theta function and Xi function with the proof of its functional equation and then it will be used to prove the analytic continuation of zeta function as well as its functional equation.

Keywords— *Riemann zeta function, Analytic continuation, Gamma function, Jacobi theta function, Xi function.*

1. INTRODUCTION

Analytic continuation means extending an analytic function defined in a domain to one defined in a larger domain [1-8].

Definition: If $f(z)$ is analytic in a domain D and $F(z)$ is analytic in a domain $D_0 \supset D$ with $F(z) = f(z)$ in D, then we say that F is an analytic continuation of f [1-12].

2. THE GAMMA FUNCTION

We start with the analytic continuation of the Riemann zeta function with respect of Gamma function [1-10, 12]. We introduce the Gamma function for $\mathrm{Re}(s) > 0$ and we will show how $\Gamma(s)$ is connected to the Riemann zeta function [1-11].

Theorem 1: Prove that

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^t - 1} dt \quad \text{for } \mathrm{Re}(s) > 1 \tag{2.1}$$

Proof: To prove the theorem by (2.1) we have

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt$$

Now put $t = nu$, so that $dt = ndu$. $(n \in \mathbb{N})$

Not that $t = 0 \Rightarrow u = 0$, $t = \infty \Rightarrow u = \infty$

Therefore

$$\Gamma(s) = \int_0^\infty e^{-nu} (nu)^{s-1} ndu$$

$$= \int_0^\infty e^{-nu} n^s u^{s-1} du$$

$$\Gamma(s) n^{-s} = \int_0^\infty e^{-nu} u^{s-1} du$$

Now taking summation from 1 to ∞, we get

$$\Gamma(s)\sum_{n=1}^{\infty} n^{-s} = \sum_{n=1}^{\infty}\int_{0}^{\infty} e^{-nu}u^{s-1}du$$

$$\Gamma(s)\sum_{n=1}^{\infty} n^{-s} = \int_{0}^{\infty}\left(\sum_{n=1}^{\infty} e^{-nu}\right)u^{s-1}du$$

$$\Gamma(s)\sum_{n=1}^{\infty} n^{-s} = \int_{0}^{\infty}\frac{e^{-u}}{1-e^{-u}}u^{s-1}du$$

We get

$$\Gamma(s)\zeta(s) = \int_{0}^{\infty}\frac{u^{s-1}}{e^{u}-1}du$$

Replacing u by t in the above equation

$$\therefore\ \zeta(s) = \frac{1}{\Gamma(s)}\int_{0}^{\infty}\frac{t^{s-1}}{e^{t}-1}dt\,.$$

Theorem 2: For $\mathrm{Re}(s) > 1$,

$$\Gamma(s)\zeta(s) = \frac{1}{s-1} - \frac{1}{2s} + \sum_{k=1}^{\infty}\frac{B_{2k}}{2k!}\frac{1}{s+2k-1} + \int_{1}^{\infty}\frac{x^{s-1}}{e^{x}-1}dx \qquad (2.2)$$

Proof: To prove the theorem we have

$$\Gamma(s) = \int_{0}^{\infty} e^{-t}t^{s-1}dt$$

First, for an integer n , this identity is easy to prove. Just change variables by $nx = t$, so $ndx = dt$
And

$$x^{s-1} = \frac{t^{s-1}}{n^{s-1}}$$, put in (2.1) we get

$$\int_{0}^{\infty} e^{-nx}x^{s-1}dx = \frac{1}{n^{s}}\Gamma(s)$$

We take summation to both sides overall $n \geq 1$, we get

$$\Gamma(s)\zeta(s) = \sum_{n=1}^{\infty}\int_{0}^{\infty} e^{-nx}x^{s-1}dx$$

Now change the sum and integral to get

$$\Gamma(s)\zeta(s) = \int_{0}^{\infty}\left\{\sum_{n=1}^{\infty} e^{-nx}\right\}x^{s-1}dx$$

Because $e^{-nx} = e^{-x^{n}}$, we see a Geometric series in the variable e^{-x}, but starting with $n = 1$. So, we have

$$\Gamma(s)\zeta(s) = \int_{0}^{\infty}\frac{e^{-x}}{1-e^{-x}}x^{s-1}dx$$

$$= \int_{0}^{\infty}\frac{1}{e^{x}-1}x^{s-1}dx$$

After multiplying numerator and denominator by e^{-x} we get

$$\Gamma(s)\zeta(s) = \int_{0}^{\infty}\frac{1}{e^{x}-1}x^{s-1}dx$$

Now we can break the integral into two pieces

$$\Gamma(s)\zeta(s) = \int_{0}^{1}\frac{1}{e^{x}-1}x^{s-1}dx + \int_{1}^{\infty}\frac{1}{e^{x}-1}x^{s-1}dx$$

We need to examine the first piece. We can write $\frac{1}{e^{x}-1}$ as a series with Bernoulli numbers we get

$$\int_0^1 \frac{1}{e^x-1}x^{s-1}dx = \int_0^1 \left\{\frac{1}{x}+\frac{1}{2}+\sum_{k=1}^{\infty}\frac{B_{2k}x^{2k-1}}{2k!}\right\}x^{s-1}dx$$

$$\int_0^1 \frac{1}{e^x-1}x^{s-1}dx = \int_0^1 \left\{x^{s-2}+\frac{x^{s-1}}{2}+\sum_{k=1}^{\infty}\frac{B_{2k}x^{2k+s-2}}{2k!}\right\}dx$$

Now integrate each term separately, we change the sum and integral.
They are all of the same forms

$$\int_0^1 x^{s+n}dx = \left(\frac{x^{s+n+1}}{s+n+1}\right)\Bigg|_0^1 = \frac{1}{s+n+1}$$

Where $n=-1,-2$ or $2k-2$.
So

$$\int_0^1 \frac{1}{e^x-1}x^{s-1}dx = \frac{1}{s-1}+\frac{1}{2s}+\sum_{k=1}^{\infty}\frac{B_{2k}}{2k!}\frac{1}{s+2k-1}$$

$$\therefore\ \Gamma(s)\zeta(s) = \frac{1}{s-1}-\frac{1}{2s}+\sum_{k=1}^{\infty}\frac{B_{2k}}{2k!}\frac{1}{s+2k-1}+\int_1^{\infty}\frac{x^{s-1}}{e^x-1}dx\,.$$

This theorem gives the analytic continuation of the function $\Gamma(s)\zeta(s)$.

3. JACOBI THETA FUNCTION

The next function to consider is the Jacobi theta function, and (inadvertently) the Poisson summation formula. We begin by explaining the Jacobi theta function, and how Fourier analysis gives us relevant properties [1-7].

Definition: For any complex number s, the Jacobi Theta function $\theta(s)$ is defined as

$$\theta(s) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 s} \tag{3.3}$$

Note that

$$\sum_{n=1}^{\infty} e^{-\pi n^2 s} = \frac{\theta(s)-1}{2} \tag{3.4}$$

To show that (3.4) we have by (3.3)

$$\theta(s) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 s}$$

$$\theta(s) = 1+\sum_{n=-\infty}^{-1} e^{-\pi n^2 s}+\sum_{n=1}^{\infty} e^{-\pi n^2 s}$$

$$\theta(s)-1 = \sum_{n=1}^{\infty} e^{-\pi(-n)^2 s}+\sum_{n=1}^{\infty} e^{-\pi n^2 s}$$

$$\theta(s)-1 = 2\sum_{n=1}^{\infty} e^{-\pi n^2 s}$$

$$\frac{\theta(s)-1}{2} = \sum_{n=1}^{\infty} e^{-\pi n^2 s}$$

The summation $\sum_{n=1}^{\infty} e^{-\pi n^2 s}$ is often called the psi function and is denoted as $\psi(s)$. Therefore, we may also write the Jacobi theta function as

$$\theta(s) = 1+2\psi(s) \tag{3.5}$$

Proposition: The functional equation of the Jacobi theta function is given [1-4], as

$$\theta(s) = \frac{1}{\sqrt{s}}\theta\left(\frac{1}{s}\right) \qquad 3.6)$$

Before proving this proposition it is necessary to introduce:

The Poisson Summation Formula

If $\sum_{-\infty}^{\infty} f(t+n)$ is uniformly converges for $0 \le t \le 1$, f is continuous and if $\sum_{-\infty}^{\infty} f(n)e^{2\pi nit}$ converges [6].

Hence

$$\sum_{-\infty}^{\infty} f(t+n) = \sum_{-\infty}^{\infty} \hat{f}(ne^{2\pi nit})$$

Where

$$\hat{f}(n) = \int_{-\infty}^{\infty} f(t)e^{-2\pi nit}dt$$

Proof: We have by (3, 3) and $f(n) = e^{-\pi n^2 s}$ we see by applying Poisson summation formula

$$\theta(s) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 s} = \sum_{k\in\mathbb{Z}} \int_{-\infty}^{\infty} e^{-\pi x^2 s} e^{-2\pi ikx} dx$$

$$= \sum_{k\in\mathbb{Z}} \int_{-\infty}^{\infty} e^{-\pi x^2 s - 2\pi ikx} dx$$

We use the Gauss integral trick to solve the integral above. Thus, we aim to from an integral that look like e^{-a^2} by completing the square for $-\pi x^2 s - 2\pi ikx$. Indeed we may write

$$\theta(s) = \sum_{k\in\mathbb{Z}} \int_{-\infty}^{\infty} e^{-\pi x^2 s - 2\pi ikx} dx$$

$$\theta(s) = \sum_{k\in\mathbb{Z}} \int_{-\infty}^{\infty} e^{-\pi s\left(x^2 + 2i\frac{k}{s}x + i^2\frac{k^2}{s^2} - i^2\frac{k^2}{s^2}\right)} dx$$

$$\theta(s) = \sum_{k\in\mathbb{Z}} \int_{-\infty}^{\infty} e^{-\pi s\left(\left(x + i\frac{k}{s}\right)^2 - i^2\frac{k^2}{s^2}\right)} dx$$

$$\theta(s) = \sum_{k\in\mathbb{Z}} \int_{-\infty}^{\infty} e^{-\pi s\left(x + i\frac{k}{s}\right)^2 + \pi s(-1)\frac{k^2}{s^2}} dx$$

$$\theta(s) = \sum_{k\in\mathbb{Z}} \int_{-\infty}^{\infty} e^{\frac{-\pi k^2}{s}} e^{-\pi s\left(x + i\frac{k}{s}\right)^2} dx$$

$$\theta(s) = \sum_{k\in\mathbb{Z}} e^{\frac{-\pi k^2}{s}} \int_{-\infty}^{\infty} e^{-\pi s\left(x + i\frac{k}{s}\right)^2} dx$$

We want to introduce a substitution, $x + i\frac{k}{s} = a$ since $i\frac{k}{s}$ is a constant, $dx = da$.

Therefore, we have

$$\theta(s) = \sum_{k\in\mathbb{Z}} e^{\frac{-\pi k^2}{s}} \int_{-\infty}^{\infty} e^{-\pi s a^2} da$$

Now we have form to use the Gauss integral trick, which implies that

$$\int_{-\infty + i\frac{k}{s}}^{\infty + i\frac{k}{s}} e^{-\pi s a^2} da = \int_{-\infty}^{\infty} e^{-\pi s a^2} da = \sqrt{\frac{\pi}{\pi s}}$$

For positive a , and therefore

$$\theta(s) = \sum_{k\in\mathbb{Z}} e^{\frac{-\pi k^2}{s}} \int_{-\infty}^{\infty} e^{-\pi s a^2} da = \sum_{k\in\mathbb{Z}} e^{\frac{-\pi k^2}{s}} \sqrt{\frac{\pi}{\pi s}}$$

$$= \sum_{k\in\mathbb{Z}} e^{\frac{-\pi k^2}{s}} \sqrt{\frac{1}{s}}$$

$$= \frac{1}{\sqrt{s}} \theta\left(\frac{1}{s}\right)$$

Note:
By (3.5) we have

$$\theta(s) = 1 + 2\psi(s) \Rightarrow \psi(s) = \frac{\theta(s)-1}{2}$$

Also, by (3.6) we have $\theta(s) = \frac{1}{\sqrt{s}}\theta\left(\frac{1}{s}\right)$ by (4.5) we get $\theta\left(\frac{1}{s}\right) = 1 + 2\psi\left(\frac{1}{s}\right)$ implies that

$$\theta(s) = \frac{1}{\sqrt{s}}\left(1 + 2\psi\left(\frac{1}{s}\right)\right)$$

Put in

$$\psi(s) = \frac{\theta(s)-1}{2} \text{ We get}$$

$$\psi(s) = \frac{1}{\sqrt{s}}\psi\left(\frac{1}{s}\right) + \frac{1}{2\sqrt{s}} - \frac{1}{2} \tag{3.7}$$

Theorem 3: For $\mathrm{Re}(s) > 1$, we have

$$\pi^{\frac{-s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \frac{1}{2}\int_0^\infty u^{\frac{s}{2}-1}(\theta(u)-1)du \tag{3.8}$$

Proof: We first the following

$$\int_0^\infty e^{-\pi n^2 u} u^{\frac{s}{2}-1} du = \pi^{\frac{-s}{2}}\Gamma\left(\frac{s}{2}\right)n^{-s} \quad \text{For } n \ge 1$$

Put $t = \pi n^2 u$, $dt = \pi n^2 du$

$$u = 0 \Rightarrow t = 0$$
$$u = \infty \Rightarrow t = \infty$$

Therefore

$$\int_0^\infty e^{-\pi n^2 u} u^{\frac{s}{2}-1} du = \int_0^\infty e^{-t}\left(\frac{t}{\pi n^2}\right)^{\frac{s}{2}-1}\frac{1}{\pi n^2}dt$$

$$\int_0^\infty e^{-\pi n^2 u} u^{\frac{s}{2}-1} du = \frac{1}{\left(\pi n^2\right)^{\frac{s}{2}}}\int_0^\infty e^{-t}(t)^{\frac{s}{2}-1}dt$$

$$\int_0^\infty e^{-\pi n^2 u} u^{\frac{s}{2}-1} du = \pi^{\frac{-s}{2}} n^{-s}\int_0^\infty e^{-t}(t)^{\frac{s}{2}-1}dt$$

$$\int_0^\infty e^{-\pi n^2 u} u^{\frac{s}{2}-1} du = \pi^{\frac{-s}{2}} n^{-s}\Gamma\left(\frac{s}{2}\right) \tag{a}$$

We now prove the main result of the theorem

$$\frac{1}{2}\int_0^\infty u^{\frac{s}{2}-1}(\theta(u)-1)du = \int_0^\infty u^{\frac{s}{2}-1}\frac{(\theta(u)-1)}{2}du$$

$$= \int_0^\infty u^{\frac{s}{2}-1}\left(\sum_{n=1}^\infty e^{-\pi n^2 u}\right)du$$

$$= \sum_{n=1}^\infty\left(\int_0^\infty u^{\frac{s}{2}-1}e^{-\pi n^2 u}du\right) \text{by (a)}$$

$$= \sum_{n=1}^\infty \pi^{\frac{-s}{2}}\Gamma\left(\frac{s}{2}\right)n^{-s}$$

$$= \pi^{\frac{-s}{2}}\Gamma\left(\frac{s}{2}\right)\sum_{n=1}^\infty n^{-s}$$

$$= \pi^{\frac{-s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) \ .$$

$$\therefore \pi^{\frac{-s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)=\frac{1}{2}\int_0^\infty u^{\frac{s}{2}-1}(\theta(u)-1)du \ .$$

4. COMPLETED RIEMANN ZETA FUNCTION

The Riemann zeta function $\zeta(s)$ can be analytically continued to the whole complex plane except at $s=1$, [1-9].
At the first define the completed zeta function $\xi(s)$ and prove some of its properties.

Definition: (Xi Function): For $\mathrm{Re}(s)>1$, the Xi Function define [1-6] by

$$\xi(s)=\pi^{\frac{-s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) \tag{4.9}$$

Note that by (3.8), we have

$$\xi(s)=\frac{1}{2}\int_0^\infty u^{\frac{s}{2}-1}(\theta(u)-1)du$$

The function $\xi(s)$ is holomorphic for $\mathrm{Re}(s)>1$and has simple poles at $s=0,1$ [1-12].
Theorem 4: We have for $\mathrm{Re}(s)>1$, Xi Function give [3] as

$$\xi(s)=\pi^{\frac{-s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

Then it can be analytically continued to the whole complex plane except when $s=0,1$, and it satisfies the functional equation

$$\xi(s)=\xi(1-s) \tag{4.10}$$

Proof: At the first to prove the theorem, we must find the equivalent of a right-hand side of (4.9)
We find

$$\Gamma\left(\frac{s}{2}\right)=\int_0^\infty t^{\frac{s}{2}-1}e^{-t}dt \tag{b}$$

Let $n\in\mathbb{N}>0$ and substitute $t=\pi n^2x$ in (1). Then $\Gamma(\frac{s}{2})$ becomes

$$\Gamma\left(\frac{s}{2}\right)=\int_0^\infty(\pi n^2x)^{\frac{s}{2}-1}e^{-\pi n^2x}\pi n^2dx$$

$$=\int_0^\infty\pi^{\frac{s}{2}-1}n^{2\left(\frac{s}{2}-1\right)}x^{\frac{s}{2}-1}e^{-\pi n^2x}\pi n^2dx$$

$$=\pi^{\frac{s}{2}}n^s\int_0^\infty x^{\frac{s}{2}-1}e^{-\pi n^2x}dx$$

Now multiplying both sides by $\dfrac{\pi^{\frac{-s}{2}}}{n^s}$ we get

$$\frac{\pi^{\frac{-s}{2}}}{n^s}\Gamma\left(\frac{s}{2}\right)=\int_0^\infty x^{\frac{s}{2}-1}e^{-\pi n^2x}dx$$

And take the summation over n,

$$\sum_{n=1}^\infty\frac{\pi^{\frac{-s}{2}}}{n^s}\Gamma\left(\frac{s}{2}\right)=\sum_{n=1}^\infty\int_0^\infty x^{\frac{s}{2}-1}e^{-\pi n^2x}dx$$

Notice that the left-hand side of the above equation is $\pi^{\frac{-s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)$.

For the right-hand side, [1-3], we can interchange the integral and summation because both are absolutely convergent for $\mathrm{Re}(s)>1$
The above expression is then equal to

$$\pi^{\frac{-s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)=\int_0^\infty\sum_{n=1}^\infty x^{\frac{s}{2}-1}e^{-\pi n^2x}dx$$

$$=\int_0^\infty x^{\frac{s}{2}-1}\sum_{n=1}^\infty e^{-\pi n^2 x}dx$$

It is observed that the summation on the right-hand side of the above equation is nothing but the ψ function, and thus

$$\xi(s)=\pi^{\frac{-s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)=\int_0^\infty x^{\frac{s}{2}-1}\psi(x)dx$$

Now we split the right-hand side into two integrals, namely

$$\xi(s)=\int_0^1 x^{\frac{s}{2}-1}\psi(x)dx+\int_1^\infty x^{\frac{s}{2}-1}\psi(x)dx$$

We compute the first integral we get

$$\int_0^1 x^{\frac{s}{2}-1}\psi(x)dx=\int_0^1 x^{\frac{s}{2}-1}\left(\frac{1}{\sqrt{x}}\psi\left(\frac{1}{x}\right)+\frac{1}{2\sqrt{x}}-\frac{1}{2}\right)dx \qquad \text{,by (3.7)}$$

$$=\int_0^1\left(x^{\frac{s-3}{2}}\psi\left(\frac{1}{x}\right)+\frac{1}{2}x^{\frac{s-3}{2}}-\frac{1}{2}x^{\frac{s}{2}-1}\right)dx$$

$$=\int_0^1 x^{\frac{s-3}{2}}\psi\left(\frac{1}{x}\right)dx+\frac{1}{2}\int_0^1\left(x^{\frac{s-3}{2}}-x^{\frac{s}{2}-1}\right)dx=\int_0^1 x^{\frac{s-3}{2}}\psi\left(\frac{1}{x}\right)dx+\frac{1}{2}\left[\frac{2}{s-1}x^{\frac{s-1}{2}}-\frac{2}{s}x^{\frac{s}{2}}\right]_0^1$$

$$=\int_0^1 x^{\frac{s-3}{2}}\psi\left(\frac{1}{x}\right)dx+\frac{1}{2}\left[\frac{2}{s-1}(1)^{\frac{s-1}{2}}-\frac{2}{s}(1)^{\frac{s}{2}}-\frac{2}{s-1}(0)^{\frac{s-1}{2}}-\frac{2}{s}(0)^{\frac{s}{2}}\right]$$

$$=\int_0^1 x^{\frac{s-3}{2}}\psi\left(\frac{1}{x}\right)dx+\frac{1}{s(s-1)}\ .$$

Furthermore, by changing the variable as $x=\frac{1}{u}$ on the right-hand side

$$x=0\Rightarrow u=\infty$$
$$x=\infty\Rightarrow u=0$$

We obtain

$$\int_0^1 x^{\frac{s-3}{2}}\psi\left(\frac{1}{x}\right)dx=\int_\infty^1\left(\frac{1}{u}\right)^{\frac{s-3}{2}}\psi(u)\left(-u^{-2}\right)du+\frac{1}{s(s-1)}$$

$$=\int_1^\infty(u)^{\frac{-s+3}{2}}\psi(u)\left(u^{-2}\right)du+\frac{1}{s(s-1)}$$

$$=\int_1^\infty(u)^{\frac{-s-1}{2}}\psi(u)du+\frac{1}{s(s-1)}$$

Hence

$$\xi(s)=\int_0^1 x^{\frac{s}{2}-1}\psi(x)dx+\int_1^\infty x^{\frac{s}{2}-1}\psi(x)dx$$

$$=\int_1^\infty x^{\frac{-s-1}{2}}\psi(x)dx+\frac{1}{s(s-1)}+\int_1^\infty x^{\frac{s}{2}-1}\psi(x)dx$$

$$=\int_1^\infty\left(x^{\frac{-s-1}{2}}+x^{\frac{s}{2}-1}\right)\psi(x)dx+\frac{1}{s(s-1)}$$

$$=\int_1^\infty\left(x^{\frac{-s+1}{2}}+x^{\frac{s}{2}}\right)x^{-1}\psi(x)dx+\frac{1}{s(s-1)} \qquad \text{(c)}$$

Now we find $\xi(1-s)$ in the right-hand side (c) substitute $s=1-s$ we get

$$\xi(1-s)=\int_1^\infty\left(x^{\frac{-1+s+1}{2}}+x^{\frac{1-s}{2}}\right)x^{-1}\psi(x)dx+\frac{1}{(1-s)(1-s-1)}$$

$$=\int_1^\infty\left(x^{\frac{s}{2}}+x^{\frac{-s+1}{2}}\right)x^{-1}\psi(x)\,dx+\frac{1}{(s-1)s}$$

$$=\xi(s) \qquad \text{by (c)}$$

$$\therefore \xi(s)=\xi(1-s).$$

Theorem 5: Prove that

$$\zeta(s)=\frac{\pi^{s-\frac{1}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)}{\Gamma\left(\frac{s}{2}\right)} \tag{4.11}$$

Proof: We know that by (4.9) we have

$$\xi(s)=\pi^{\frac{-s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) \tag{d}$$

And we know that by (4.10) we have

$$\xi(s)=\xi(1-s) \tag{e}$$

Expand (e) with respect (d) we see

$$\pi^{\frac{-s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)=\pi^{\frac{-(1-s)}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)$$

$$\zeta(s)=\frac{\pi^{\frac{s}{2}}\pi^{\frac{-(1-s)}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)}{\Gamma\left(\frac{s}{2}\right)}$$

$$\therefore \zeta(s)=\frac{\pi^{s-\frac{1}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)}{\Gamma\left(\frac{s}{2}\right)}.$$

Theorem 6: For any complex number $s\neq 1$,[8] we have

$$\zeta(1-s)=2(2\pi)^{-s}\Gamma(s)\cos\left(\frac{\pi s}{2}\right)\zeta(s) \tag{4.12}$$

Or, equivalently,

$$\zeta(s)=2(2\pi)^{s-1}\Gamma(1-s)\sin\left(\frac{\pi s}{2}\right)\zeta(1-s) \tag{4.13}$$

Proof: To prove the theorem using (3.7) and (3.9), change variable $s=\frac{s}{2}$ in (4.9), we get

$$\Gamma(s)=\frac{2^{s-1}}{\sqrt{\pi}}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right) \tag{f}$$

Therefore, we have

$$\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)=\frac{\sqrt{\pi}}{2^{s-1}}\Gamma(s) \tag{g}$$

Also, substitute $s=\frac{s+1}{2}$ in (3.7) we see that

$$\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(1-\frac{s+1}{2}\right)=\frac{\pi}{\sin\left(\frac{\pi s}{2}+\frac{\pi}{2}\right)} \tag{h}$$

Since $\frac{1}{\Gamma(s)}$ is an entire function, we divide equation (g) by equation (h) to get

$$\frac{\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{1+s}{2}\right)}{\Gamma\left(\frac{1+s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)}=\frac{\frac{\sqrt{\pi}}{2^{s-1}}\Gamma(s)}{\frac{\pi}{\sin\left(\pi\left(\frac{s+1}{2}\right)\right)}}$$

$$\frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)}=\frac{2}{2^{s}\sqrt{\pi}}\cos\left(\frac{\pi s}{2}\right)\Gamma(s) \tag{i}$$

Also, by (4.11) we have

$$\pi^{\frac{-s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)=\pi^{\frac{-(1-s)}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)$$

$$\zeta(1-s)=\pi^{\frac{1}{2}-s}\frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)}\zeta(s) \tag{j}$$

Put (i) in (g) we get

$$\therefore \zeta(1-s)=2(2\pi)^{-s}\Gamma(s)\cos\left(\frac{\pi s}{2}\right)\zeta(s)$$

In particular, replace $s=1-s$ in (4.12) we get the result (4.13)

$$\therefore \zeta(s)=2(2\pi)^{s-1}\Gamma(1-s)\sin\left(\frac{\pi s}{2}\right)\zeta(1-s).$$

5. CONCLUSION

This dissertation has demonstrated the most important aspects of the zeta function such as its analytic continuation, proved the functional equation and introduced its applications. Additionally, some of interesting properties of the zeta function have been proved. Moreover, the zeros of $\zeta(s)$ have been discussed particularly, trivial zeros, nontrivial zeros and the fact that $\zeta(s)$ has no zero when $Re(s) > 1$. Following this dissertation, many other parts of the Riemann zeta function and its analytic continuation remain to be studied. One suggestion for interested researchers is to look at the famous Riemann Hypothesis, while another would be to look at convexity as it applies to the Riemann zeta function.

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