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A two-parameter fourth-order methods for solving nonlinear equations

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ABSTRACT

In this paper, we present two-parameter iterative methods for solving nonlinear equations. Convergence analysis shows that the new methods converge of order four. Some illustrative examples are solved to show the validity of our method.

Keywords— *Simple roots, Nonlinear equations, Order of convergence*

1. INTRODUCTION

We consider iterative methods to find a simple root α, i.e., $f(\alpha)=0$ and $f'(\alpha)\neq 0$, of a nonlinear equation, $f(x)=0$ that uses f and f' but not the higher derivatives of f. The best-known iterative method for the calculation of α is Newton's method defined by

$$x_{n+1}=x_n-\frac{f(x_n)}{\acute{f}(x_n)} \tag{1}$$

Where x_0 is an initial approximation sufficiently close to α. This method is quadratically convergent [10]. There exists a modification of Newton's method with third-order convergence due to Potra and Ptak [11], defined by

$$x_{n+1}=x_n-\frac{f(x_n)+f'(x_n-\frac{f(x_n)}{\acute{f}(x_n)})}{f'(x_n)} \tag{2}$$

Recently, some Newton-type methods have been developed by [1-9, 12, 13]. To obtain some of those iterative methods, the Adomian decomposition method was applied in [1, 3], He's homotopy perturbation method in [2, 7] and Liao's homotopy analysis method in [4]. Some of the other methods have been derived by considering different quadrature formulas for the computation of the integral arising from Newton's theorem:

$$\mathrm{f}(\mathrm{x})=\mathrm{f}(\mathrm{x}_n)+\int_{\mathrm{x}_n}^{\mathrm{x}}\mathrm{f}'(\mathrm{t})\mathrm{dt}. \tag{3}$$

Weerakoon and Fernando [13] applied the rectangular and trapezoidal rules to the integral of (3) to rederive the Newton method and arrive at the cubically convergent method

$$x_{n+1}=x_n-\frac{2f(x_n)}{f'(x_n)+f'\left(x_n-\frac{f(x_n)}{f'(x_n)}\right)} \tag{4}$$

While Frontini and Sormani [6] obtained the cubically convergent method

$$x_{n+1}=x_n-\frac{f(x_n)}{f'\left(x_n-\frac{f(x_n)}{2f'(x_n)}\right)} \tag{5}$$

By considering the midpoint rule. In [8], Homeier derived the following cubically convergent iteration scheme:

$$x_{n+1}=x_n-\frac{f(x_n)}{2}\left(\frac{1}{f'(x_n)}+\frac{1}{f'\left(x_n-\frac{f(x_n)}{f'(x_n)}\right)}\right) \tag{6}$$

By considering Newton's theorem for the inverse function $x = f(y)$ instead of $y = f(x)$, recently, Kou, *et. al* in [11] considered Newton's theorem on a new interval of integration and arrived at the following cubically convergent iterative scheme

$$x_{n+1} = x_n - \frac{f\left(x_n + \frac{f(x_n)}{f'(x_n)}\right) - f(x_n)}{f'(x_n)} \tag{7}$$

The aforementioned methods require three functional evaluations of the given function and its first derivative, but no evaluations of the second or higher derivatives. Finding the iterative methods with third-order convergence, not requiring the computation of second derivatives is important and interesting from the practical point of view and becomes active now. In this paper, we present a new two-parameter family of modified Newton's methods that do not require the computation of second-order derivatives of the function. Derivation of the family is based on finding a correction term for the second substep in the two-substep Newton method, which will be described in the following section. We prove that each family member is a third-order convergent. In particular, we show that the Potra and Ptak third-order method can be obtained as a special case of the new family. Finally, the comparison with other third-order methods is given to illustrate the performance of the presented methods.

2. ITERATIVE METHODS AND CONVERGENCE ANALYSIS

We try to derive some new methods in a similar manner of [5] as follows: consider the two-substep Newton-Raphson method given by

$$z_n = x_n - \frac{f\left(x_n + \frac{f(x_n)}{f'(x_n)}\right) - f(x_n)}{f'(x_n)} \tag{8}$$

$$x_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)} \tag{9}$$

Our aim is to find a correction term for the second sub step (9) that will yield a third-order method. To do this,first consider fitting the function f(x) around the point $(x_n, f(x_n))$ with the third-degree polynomial

$$g(x) = ax^3 + bx^2 + cx + d \tag{10}$$

Imposing the tangency condition at the nth iterate x_n

$$g'(x_n) = f'(x_n), \tag{11}$$

On to (10), we have

$$c = f'(x_n) - 3ax_n^2 - 2bx_n \tag{12}$$

Thereby obtaining the first derivative of the approximating polynomial

$$g'(x) = 3ax^2 + 2bx + f'(x_n) - 3ax_n^2 - 2bx_n \tag{13}$$

Now, when z_n is defined by (8), we approximate $f'(z_n)$ as

$$f'(z_n) \approx g'(z_n) = \left(\left(f'(x_n)\right)^2 - (6ax_n + 2b)f(A) + (6ax_n + 2b)f(x_n) + \frac{3a\left(f(A)\right)^2}{f'(x_n)} - \frac{6af(A)f(x_n)}{f'(x_n)} + \frac{3a\left(f(x_n)\right)^2}{f'(x_n)}\right)\frac{1}{f'(x_n)} \tag{14}$$

where $A = \frac{x_n f'(x_n) - f(x_n)}{f'(x_n)}$

Using (14) in (9) we obtain the new two-parameter family of methods

$$z_n = x_n - \frac{f\left(x_n + \frac{f(x_n)}{f'(x_n)}\right) - f(x_n)}{f'(x_n)} \tag{15}$$

$$x_{n+1} = z_n - \frac{f(z_n)}{(\beta_1 + \beta_2)\frac{1}{f'(x_n)}}$$

where

$$\beta_1 = (f'(x_n))^2 - (6ax_n + 2b)f(A) + (6ax_n + 2b)f(x_n),$$

$$\beta_2 = \frac{3a\left(f(A)\right)^2}{f'(x_n)} - \frac{6af(A)f(x_n)}{f'(x_n)} + \frac{3a\left(f(x_n)\right)^2}{f'(x_n)} \tag{16}$$

z_n is defined by (15) and $A = \frac{x_n f'(x_n) - f(x_n)}{f'(x_n)}$.

The family (16) includes, as particular cases, the following ones:
For $a = 0$, $b = 0$, we obtain:

$$x_{n+1} = z_n - \frac{f(z_n)}{f'(x_n)}, \tag{17}$$

where z_n is defined by (15).

For $a = 0, b = 1$, we obtain a new fourth-order method:

$$x_{n+1} = z_n - \frac{f(z_n)}{\left(\left(f'(x_n)\right)^2 - 2f(A) + 2f(x_n)\right)\frac{1}{f'(x_n)}} \tag{18}$$

where z_n is defined by (15) and $A = \frac{x_n f'(x_n) - f(x_n)}{f'(x_n)}$
For $a = 1$, $b = 0$,
we obtain another new fourth-order method:

$$x_{n+1} = z_n - \frac{f(z_n)}{(\gamma_1 + \gamma_2)\frac{1}{f'(x_n)}} \tag{19}$$

Where,

$$\gamma_1 = \left(f'(x_n)\right)^2 - 6x_n f(A) + 6x_n f(x_n)$$

$$\gamma_2 = \frac{3\left(f(A)\right)^2}{f'(x_n)} - \frac{6f(A)f(x_n)}{f'(x_n)} + \frac{3\left(f(x_n)\right)^2}{f'(x_n)}$$

z_n is defined by (15) and $A = \frac{x_n f'(x_n) - f(x_n)}{f'(x_n)}$.

3. CONVERGENCE ANALYSIS

In this section, we try to find the order of convergence of our new iterative method by using Maple. In this section, we consider the convergence analysis of the iterative technique given by equations (15)-(16) by the following theorem using Maple:

Theorem 1:
Assume that the function $f: I \to R$ for an open interval I has a simple root $\alpha \in I$. If $f(x)$ is sufficiently smooth on the neighborhood of the root α, then the iterative method defined by equations (15)-(16) converges of order four.

Proof:

$> v := taylor(f(x_n), x_n = a, 5);$

$$v := f(a) + D(f)(a)(x_n - a) + \frac{1}{2}D^{(2)}(f)(a)(x_n - a)^2 + \frac{1}{6}D^{(3)}(f)(a)(x_n - a)^3 + \frac{1}{24}D^{(4)}(f)(a)(x_n - a)^4 + O((x_n - a)^5)$$

$> q_2 := algsubs(f(a) = 0, v);$

$$q_2 := D(f)(a)(x_n - a) + \frac{1}{2}D^{(2)}(f)(a)(x_n - a)^2 + \frac{1}{6}D^{(3)}(f)(a)(x_n - a)^3 + \frac{1}{24}D^{(4)}(f)(a)(x_n - a)^4 + O((x_n - a)^5)$$

$> u := \frac{d}{dx_n}(taylor(f(x_n), x_n = a, 5);$

$$u := D(f)(a) + D^{(2)}(f)(a)(x_n - a) + \frac{1}{2}D^{(3)}(f)(a)(x_n - a)^2 + \frac{1}{6}D^{(4)}(f)(a)(x_n - a)^3 + O((x_n - a)^4)$$

$> o := expand\left(taylor\left(x_n + \frac{q_2}{u}, x_n = a\right)\right);$

$$o := a + 2\,(x_n - a) - \frac{1}{2}\frac{D^{(2)}(f)(a)}{D(f)(a)}(x_n - a)^2 + \left(-\frac{1}{3}\frac{D^{(3)}(f)(a)}{D(f)(a)} + \frac{1}{2}\frac{D^{(2)}(f)(a)^2}{D(f)(a)^2}\right)(x_n - a)^3 + \left(-\frac{1}{8}\frac{D^{(4)}(f)(a)}{D(f)(a)} + \frac{7}{12}\frac{D^{(2)}(f)(a)\,D^{(3)}(f)(a)}{D(f)(a)^2} - \frac{1}{2}\frac{D^{(2)}(f)(a)^3}{D(f)(a)^3}\right)(x_n - a)^4 + O((x_n - a)^5)$$

$> p := expand\big(taylor(f(o), x_n = a)\big);$

$$p := f(a) + 2\,D(f)(a)(x_n - a) + \frac{3}{2}D^{(2)}(f)(a)(x_n - a)^2 + \left(D^{(3)}(f)(a) - \frac{1}{2}\frac{D^{(2)}(f)(a)^2}{D(f)(a)}\right)(x_n - a)^3 + \left(\frac{13}{24}D^{(4)}(f)(a) - \frac{13}{12}\frac{D^{(2)}(f)(a)\,D^{(3)}(f)(a)}{D(f)(a)} + \frac{5}{8}\frac{D^{(2)}(f)(a)^3}{D(f)(a)^2}\right)(x_n - a)^4 + O((x_n - a)^5)$$

$> d := algsubs(f(a) = 0, p);$

$$d := 2\,D(f)(a)(x_n - a) + \frac{3}{2}D^{(2)}(f)(a)(x_n - a)^2 + \left(D^{(3)}(f)(a) - \frac{1}{2}\frac{D^{(2)}(f)(a)^2}{D(f)(a)}\right)(x_n - a)^3 + \left(\frac{13}{24}D^{(4)}(f)(a) - \frac{13}{12}\frac{D^{(2)}(f)(a)\,D^{(3)}(f)(a)}{D(f)(a)} + \frac{5}{8}\frac{D^{(2)}(f)(a)^3}{D(f)(a)^2}\right)(x_n - a)^4 + O((x_n - a)^5)$$

$> y := expand\left(taylor\left(x_n - \frac{d-q_2}{u}, x_n = a\right)\right);$

$$y := a + \left(-\frac{1}{3}\frac{D^{(3)}(f)(a)}{D(f)(a)} + \frac{1}{2}\frac{D^{(2)}(f)(a)^2}{D(f)(a)^2}\right)(x_n - a)^3 + \left(-\frac{1}{3}\frac{D^{(4)}(f)(a)}{D(f)(a)} + \frac{17}{2}\frac{D^{(2)}(f)(a)\,D^{(3)}(f)(a)}{D(f)(a)^2} - \frac{D^{(2)}(f)(a)^3}{D(f)(a)^3}\right)(x_n - a)^4 + O((x_n - a)^5)$$

$> g := expand(taylor(f(y), x_n = a));$

$$g := f(a) + \left(-\frac{1}{3}D^{(3)}(f)(a) + \frac{1}{2}\frac{D^{(2)}(f)(a)^2}{D(f)(a)}\right)(x_n - a)^3 + \left(-\frac{1}{3}D^{(4)}(f)(a) + \frac{17}{2}\frac{D^{(2)}(f)(a)\,D^{(3)}(f)(a)}{D(f)(a)} - \frac{9}{8}\frac{D^{(2)}(f)(a)^3}{D(f)(a)^2}\right)(x_n - a)^4 + O((x_n - a)^5)$$

$> q_1 := algsubs(f(a) = 0, g);$

$$q_1 := \left(-\frac{1}{3}D^{(3)}(f)(a) + \frac{1}{2}\frac{D^{(2)}(f)(a)^2}{D(f)(a)}\right)(x_n - a)^3 + \left(-\frac{1}{3}D^{(4)}(f)(a) + \frac{17}{2}\frac{D^{(2)}(f)(a)\,D^{(3)}(f)(a)}{D(f)(a)} - \frac{9}{8}\frac{D^{(2)}(f)(a)^3}{D(f)(a)^2}\right)(x_n - a)^4 + O((x_n - a)^5)$$

$> q_4 := expand\left(taylor\left(f\left(\frac{x_n \cdot u - q_2}{u}\right), x_n = a\right)\right);$

$$q_4 := f(a) + \frac{1}{2}D^{(2)}(f)(a)(x_n - a)^2 + \left(\frac{1}{3}D^{(3)}(f)(a) - \frac{1}{2}\frac{D^{(2)}(f)(a)^2}{D(f)(a)}\right)(x_n - a)^3 + O((x_n - a)^4)$$

$> q_5 := algsubs(f(a) = 0, q_4);$

$$q_5 := \frac{1}{2}D^{(2)}(f)(a)(x_n - a)^2 + \left(\frac{1}{3}D^{(3)}(f)(a) - \frac{1}{2}\frac{D^{(2)}(f)(a)^2}{D(f)(a)}\right)(x_n - a)^3 + O((x_n - a)^4)$$

$$s_1 := u^2 - (6 \cdot c_1 \cdot x_n + 2 \cdot c_2) \cdot q_5 + (6 \cdot c_1 \cdot x_n + 2 \cdot c_2) \cdot q_2$$

$$s2 := \frac{3 \cdot c_1 \cdot (q_5)^2}{u} - \frac{6 \cdot c_1 \cdot q_5 \cdot q_2}{u} + \frac{3 \cdot c_1 \cdot (q_2)^2}{u}$$

$> k := expand\left(taylor\left(y - \frac{q_1 \cdot u}{s1 + s2}, x_n = a\right)\right)$

$$k := a + \left(-\frac{1}{3}\frac{D^{(2)}(f)(a)\,D^{(3)}(f)(a)}{D(f)(a)^2} + \frac{1}{2}\frac{D^{(2)}(f)(a)^3}{D(f)(a)^3} - \frac{2\,D^{(3)}(f)(a)\,c_1\,a}{D(f)(a)^2} - \frac{2}{3}\frac{D^{(3)}(f)(a)\,c_2}{D(f)(a)^2} + \frac{3\,D^{(2)}(f)(a)^2\,c_1\,a}{D(f)(a)^3} + \frac{D^{(2)}(f)(a)^2\,c_2}{D(f)(a)^3}\right)(x_n - a)^4 + O((x_n - a)^5)$$

Where, $k = x[n+1]$;

4. NUMERICAL EXAMPLES

All computations were done using MAPLE with 64 digit floating point arithmetics (Digits = 64). We accept an approximate solution rather than the exact root, depending on the precision (ε) of the computer. We use the following stopping criteria for computer programs: (i) $|x_{n+1} - x_n| < \varepsilon$, (ii) $|f(x_{n+1})| < \varepsilon$, and so, when the stopping criterion is satisfied, x_{n+1} is taken as the exact root α computed. We used $\varepsilon = 10^{-15}$. We present some numerical test results for various cubically convergent iterative schemes in Table 1. Compared were the Newton method (NM), the method of Weerakoon and Fernando (4) (WF), the method derived from midpoint rule (5) (MP), the method of Homeier (6) (HM), the method of Kou et al. (7) (KM), and the methods (18) (CM1) and (19) (CM2) introduced in this paper. We remark that chosen for comparison are only the methods which do not require the computation of second or higher derivatives of the function to carry out iterations. We used the following test functions:

$f_1(x) = x^3 + 4x^2 - 10,$
$f_2(x) = sin^2(x) - x^2 + 1,$
$f_3(x) = x^2 - e^x - 3x + 2,$
$f_4(x) = \cos(x) - x,$
$f_5(x) = (x-1)^3 - 1,$
$f_6(x) = \sin(x) - \frac{x}{2},$
$f_7(x) = xe^{x^2} - sin^2(x) + 3\cos(x) + 5.$

As convergence criterion, it was required that the distance of two consecutive approximations d for the zero was less than 10^{-15} Also displayed are the number of iterations to approximate the zero (IT), the approximate zero x_{n+1}, and the value

$f(x_{n+1})$. Note that the approximate zeroes were displayed only up to the 28th decimal places, so it making all looking the same though they may in fact differ. It is clear from table 1 that the proposed methods in this work show at least equal performance as compared with the other known methods of the same order. It is also seen from these numerical experiments that,

Table 1: Comparison of various fourth convergent iterative methods and the Newton method

$f_1, x_0 = 1.27$	it	x_{n+1}	$f(x_{n+1})$
NM	5	1.3652300134140968457608068290	2.70e^{-41}
WF	4	1.3652300134140968457608068290	0
MP	4	1.3652300134140968457608068290	0
HM	3	1.3652300134140968457608068290	-4.45e^{-48}
KM	4	1.3652300134140968457608068290	0
CM1	2	1.3652300134140968457608068290	-8.10^{-9}
CM2	2	1.3652300134140968457608068290	-8.10^{-9}

$f_2, x_0 = 3.5$	it	x_{n+1}	$f(x_{n+1})$
NM	7	1.4044916482153412260350868178	-3.03e^{-43}
WF	5	1.4044916482153412260350868178	-2.0e^{-63}
MP	5	1.4044916482153412260350868178	-4.56e^{-61}
HM	5	1.4044916482153412260350868178	-2.0e^{-63}
KM	5	1.4044916482153412260350868178	1.18e^{-45}
CM1	3	1.4044916482153412260350868178	1.0 10^{-9}
CM2	4	1.4044916482153412260350868178	1.010^{-9}

$f_3, x_0 = 0$	it	x_{n+1}	$f(x_{n+1})$
NM	5	0.25753028543986076045536730494	1.56e^{-49}
WF	4	0.25753028543986076045536730494	1.0e^{-63}
MP	3	0.25753028543986076045536730494	2.07e^{-55}
HM	4	0.25753028543986076045536730494	1.0e^{-63}
KM	4	0.25753028543986076045536730494	1.0e^{-63}
CM1	2	0.25753028553986076045536730494	0
CM2	2	0.25753028563986076045536730494	0

$f_4, x_0 = 1.2$	it	x_{n+1}	$f(x_{n+1})$
NM	5	0.73908513321516064165531208767	-1.90e^{-35}
WF	4	0.73908513321516064165531208767	0
MP	4	0.73908513321516064165531208767	0
HM	4	0.73908513321516064165531208767	0
KM	4	0.73908513321516064165531208767	-6.07e^{-57}
CM1	2	0.73908513321516064165531208767	0
CM2	3	0.73908513321516064165531208767	0

$f_5, x_0 = 1.8$	it	x_{n+1}	$f(x_{n+1})$
NM	6	2	2.87e^{-41}
WF	4	2	-4.01e^{-49}
MP	4	2	-7.98e^{-54}
HM	4	2	0
KM	4	2	-1.56e^{-45}
CM1	2	2	0
CM2	3	2	0

$f_6, x_0 = 13$	it	x_{n+1}	$f(x_{n+1})$
NM		Di	
WF	6	1.8954942670339809471440357381	1.63e^{-60}
MP	5	1.8954942670339809471440357381	-3.0e^{-64}
HM		Di	
KM		Di	
CM1	3	-1.8954942670339809471440357381	0
CM2	4	-1.8954942670339809471440357381	0

$f_7, x_0 = -2$	It	x_{n+1}	$f(x_{n+1})$
NM	9	-1.2076478271309189270094167584	-2.27e^{-40}
WF	7	-1.2076478271309189270094167584	-4.0e^{-63}

MP	6	-1.207647827130918927009416758 4	-4.0e^{-63}
HM	6	-1.2076478271309189270094167584	-4.0e^{-63}
KM	6	-1.2076478271309189270094167584	-4. e^{-63}
CM1	4	-1.2076478271309189270094167584	3.10^{-9}
CM2	4	-1.2076478271309189270094167584	3.10^{-9}

5. CONCLUSION

In this paper, we presented a new two-parameter family of modified Newton's methods. Per iteration, the new methods require two evaluations of the function and one evaluation of its first derivative. We have shown that each family member is cubically convergent, and in almost all of the cases, the presented methods appear to be as robust as compared other methods.

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