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Fourier series expansions of sum and products of Bernoulli functions

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ABSTRACT

We can derive from a polynomial identity, in turn, follows from the Fourier series expansion of sums of products of Bernoulli functions. We consider three types of functions given by sums of products of higher– order Bernoulli functions and derive their Fourier series expansions. Moreover, we express each of them in terms of Bernoulli functions.

Keywords— *Fourier series, Sums of products of higher-order Bernoulli functions*

1. INTRODUCTION

The three types of sums of products of higher – order Bernoulli functions in Fourier series expansions for them. Moreover, we will express them in terms of Bernoulli functions.

Let r, s be positive integers.

$$1)\ \alpha_m(\langle x\rangle) = \sum_{k=0}^{m} B_k^{(r)}(\langle x\rangle)\, B_{m-k}^{(s)}(\langle x\rangle)\ \ (m \geq 1)$$

$$2)\ \beta_m(\langle x\rangle) = \sum_{k=0}^{m} \frac{1}{k!\,(m-k)!}\, B_k^{(r)}(\langle x\rangle) B_{m-k}^{(s)}(\langle x\rangle)\ \ (m \geq 1)$$

$$3)\ \gamma_m(\langle x\rangle) = \sum_{k=1}^{m-1} \frac{1}{k(m-k)}\, B_k^{(r)}(\langle x\rangle) B_{m-k}^{(s)}(\langle x\rangle)\ \ \ (m \geq 2)$$

There are three types of Bernoulli functions.

Definition: Let r be a non-negative integer. Then the Bernoulli polynomials $B_n^{(r)}(x)$ of order r is given by the generating function

$$\left(\frac{t}{e^t-1}\right)^r e^{xt} = \sum_{m=0}^{\infty} B_m^{(r)}(x) \frac{t^m}{m!}$$

When x = 0, $B_m^{(r)} = B_m^{(r)}(0)$ are called the Bernoulli numbers of order r.
In particular, $B_m(x) = B_m^{(1)}(x)$ are called the ordinary Bernoulli polynomials.

2. THE FUNCTION $\alpha_m(\langle x\rangle)$

Let,

$$\alpha_m(\langle x\rangle) = \sum_{k=0}^{m} B_k^{(r)}(x) B_{m-k}^{(s)}(x)\ (m \geq 1)$$

We consider the function:

$$\alpha_m(\langle x\rangle) = \sum_{k=0}^{m} B_k^{(r)}(x) B_{m-k}^{(s)}(x)\ (m \geq 1) \tag{1.1}$$

defined on ℝ. Which is periodic with period 1.

The Fourier series of $\alpha_m(\langle x\rangle)$ is:

$$\sum_{n=-\infty}^{\infty} A_n^{(m)}\ e^{2\pi inx} \tag{1.2}$$

Where

$$A_n^{(m)} = \int_0^1 \alpha_m(\langle x\rangle)\, e^{-2\pi inx} dx$$
$$= \int_0^1 \alpha_m(x)\, e^{-2\pi inx} dx. \tag{1.3}$$

To proceed, observe the following:

$$\alpha'_m(x) = \sum_{k=0}^{m}\left(kB_{k-1}^{(r)}(x)B_{m-k}^{(s)}(x) + (m-k)B_k^{(r)}(x)\, B_{m-k-1}^{(s)}(x)\right)$$
$$= \sum_{k=1}^{m} k\, B_{k-1}^{(r)}(x)B_{m-k}^{(s)}(x) + \sum_{k=0}^{m-1}(m-1)\, B_k^{(r)}(x)B_{m-k-1}^{(s)}(x)$$
$$= \sum_{k=0}^{m-1}(k+1)\, B_k^{(r)}(x)B_{m-1-k}^{(s)}(x) + \sum_{k=0}^{m-1}(m-k)\, B_k^{(r)}(x)B_{m-1-k}^{(s)}(x)$$
$$= (m+1)\sum_{k=0}^{m-1} B_k^{(r)}(x)B_{m-1-k}^{(s)}(x)$$
$$= (m+1)\alpha_m(x) \tag{1.4}$$

From this, we obtain

$$\left(\frac{\alpha_{m+1}(x)}{m+2}\right)' = \alpha_m(x) \tag{1.5}$$

and,

$$\int_0^1 \alpha_m(x)dx = \frac{1}{m+2}(\alpha_{m+1}(1) - \alpha_{m+1}(0)). \tag{1.6}$$

For $m \geq 1$, we set

$$\Delta_m = \alpha_m(1) - \alpha_m(0)$$
$$= \sum_{k=0}^{m}\left(B_k^{(r)}(1)\, B_{m-k}^{(s)}(1) - B_k^{(r)}\, B_{m-k}^{(s)}\right)$$
$$= \sum_{k=0}^{m}\left(\left(B_k^{(r)} + k\, B_{k-1}^{(r-1)}\right)\left(B_{m-k}^{(s)} + (m-k)B_{m-k-1}^{(s-1)}\right) - B_k^{(r)}B_{m-k}^{(s)}\right)$$
$$= \sum_{k=0}^{m}\left((m-k)B_k^{(r)}B_{m-k-1}^{(s-1)} + kB_{k-1}^{(r-1)}B_{m-k}^{(s)} + k(m-k)B_{k-1}^{(r-1)}B_{m-k-1}^{(s-1)}\right). \tag{1.7}$$

Now, we have

$$\alpha_m(1) = \alpha_m(0) \Leftrightarrow \Delta_m = 0 \tag{1.8}$$

and

$$\int_0^1 \alpha_m(x)dx = \frac{1}{m+2}\,\Delta_{m-1} \tag{1.9}$$

to determine the Fourier coefficients $A_n^{(n)}$.
Case 1:

$$n \neq 0$$

We have,

$$A_n^{(m)} = \int_0^1 \alpha_m(x)e^{-2\pi inx} dx$$
$$= -\frac{1}{2\pi in}[\alpha_m(x)e^{-2\pi inx}]_0^1 + \frac{1}{2\pi in}\int_0^1 \alpha'_m(x)\, e^{-2\pi inx} dx$$
$$= -\frac{1}{2\pi in}\left(\alpha_m(1) - \alpha_m(0)\right) + \frac{m+1}{2\pi in}\int_0^1 \alpha_{m-1}(x)e^{-2\pi inx} dx$$
$$= \frac{m+1}{2\pi in}A_n^{(m-1)} - \frac{1}{2\pi in}\Delta_m, \tag{1.10}$$

from which by induction on m,

Thus we show that,

$$A_n^{(m)} = -\frac{1}{m+2}\sum_{j=1}^{m}\frac{(m+2)_j}{(2\pi in)^j}\,\Delta_{m-j+1} \tag{1.11}$$

Case 2:

n = 0.

We have,

$$A_0^{(m)} = \int_0^1 \alpha_m\,(x)dx = \frac{1}{m+2}\,\Delta_{m+1} \tag{1.12}$$

$\alpha_m(\langle x\rangle)\ (m \geq 1)$ is piecewise C^∞. Moreover, $\alpha_m(\langle x\rangle)$ is continuous for those integers m > s with $\Delta_m = 0$, and discontinuous with jump discontinuous at integers for those integers m > s with $\Delta_m \neq 0$.

Assume first that $\Delta_m = 0$, for an integer m.
Then $\alpha_m(0) = \alpha_m(1)$.

Hence $\alpha_m(\langle x\rangle)$ is piecewiseC^∞, and continuous.

Thus the Fourier series of $\alpha_m(\langle x\rangle)$ converges uniformly to $\alpha_m(\langle x\rangle)$, and

$$\begin{aligned}\alpha_m(\langle x\rangle) &= \frac{1}{m+2}\,\Delta_{m+1} \\ &+ \sum_{\substack{n=-\infty,\\ n\neq 0}}^{\infty}\left(-\frac{1}{m+2}\sum_{j=1}^{m}\frac{(m+2)_j}{(2\pi in)^j}\,\Delta_{m-j+1}\right)e^{2\pi inx} \\ &= \frac{1}{m+2}\,\Delta_{m+1} + \frac{1}{m+2}\sum_{j=1}^{m}\binom{m+2}{j}\,\Delta_{m-j+1}\times\left(-j!\sum_{\substack{n=-\infty,\\ n\neq 0}}^{\infty}\frac{e^{2\pi inx}}{(2\pi in)^j}\right) \\ &= \frac{1}{m+2}\,\Delta_{m+1} + \frac{1}{m+2}\sum_{j=2}^{m}\binom{m+2}{j}\,\Delta_{m-j+1}\,B_j(\langle x\rangle) \\ &\times\ \Delta_m \times \begin{cases} B_1(\langle x\rangle), & for\ x \notin \mathbb{Z}, \\ 0, & for\ x\ \in\ \mathbb{Z}. \end{cases}\end{aligned} \tag{1.13}$$

Theorem 2.1
For each positive integer l,
We let,

$$\Delta_l = \sum_{k=0}^{l}\left((l-k)B_k^{(r)}\,B_{l-k-1}^{(s-1)} + kB_{k-1}^{(r-1)}\,B_{l-k}^{(s)} + k\,(l-k)B_{k-1}^{(r-1)}B_{l-k-1}^{(s-1)}\right)\ldots B_{c_r}G_{j_1}\ldots G_{j_s}.$$

Assume that $\Delta_m = 0$, for a positive integer m.

Then we have the following:
a)

$$\sum_{k=0}^{m} B_k^{(r)}\,(\langle x\rangle)B_{m-k}^{(s)}\,(\langle x\rangle)$$

has the Fourier series expansion

$$\begin{aligned}&\sum_{k=0}^{m} B_k^{(r)}\,(\langle x\rangle)B_{m-k}^{(s)}\,(\langle x\rangle) \\ &= \frac{1}{m+2}\,\Delta_{m-1} \\ &+ \sum_{\substack{n=-\infty,\\ n\neq 0}}^{\infty}\left(-\frac{1}{m+2}\sum_{j=1}^{m}\frac{(m+2)_j}{(2\pi in)^j}\,\Delta_{m-j+1}\right)e^{2\pi inx}\end{aligned} \tag{1.14}$$

for all $x \in \mathbb{R}$, where the convergence is uniform.

b)

$$\begin{aligned}&\sum_{k=0}^{m} B_k^{(r)}\,(\langle x\rangle)B_{m-k}^{(s)}\,(\langle x\rangle) \\ &= +\frac{1}{m+2}\sum_{\substack{j=0,\\ j\neq 1}}^{m}\binom{m+2}{j}\Delta_{m-j+1}\,B_j(\langle x\rangle)\end{aligned} \tag{1.15}$$

for all $x \in \mathbb{R}$.

Assume that $\Delta_m \neq 0$, for a positive integer m.
Then $\alpha_m(0) \neq \alpha_m(1)$. Hence, $\alpha_m(\langle x \rangle)$ is piecewise C^∞, and discontinuous with jump discontinuities at integers.

The Fourier series of $\alpha_m(\langle x \rangle)$ converges point wise to $\alpha_m(\langle x \rangle)$, for $x \notin \mathbb{Z}$, and converges to

$$\frac{1}{2}\left(\alpha_m(0) + \alpha_m(1)\right) = \alpha_m(0) + \frac{1}{2}\Delta_m \tag{1.16}$$

for $x \in \mathbb{Z}$.

3. CONCLUSION

We considered the Fourier series expansion of the sum and products of Bernoulli functions of $\alpha_m(x)$. We obtain the extending by periodicity of period the Bernoulli polynomials on [0, 1]. The Fourier series are explicitly determined.

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