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Pochhammer symbol of ultra gamma function and its applications to hypergeometric functions

Anita
anita.neetu89@yahoo.com
Jai Narain Vyas University, Jodhpur, Rajasthan

K. S. Gehlot
drksgehlot@rediffmail.com
Jai Narain Vyas University, Jodhpur, Rajasthan

Chena Ram
crc2007@rediffmail.com
Jai Narain Vyas University, Jodhpur, Rajasthan

ABSTRACT

The aim of this paper is to investigate an extension of generalized hypergeometric function $_rF_s$ with r numerator and s denominator parameters with help of ultra gamma function. Some Recurrence relation of the Pochhammer symbol of Ultra Gamma Function is investigated. Certain particular cases of the derived results are considered and indicated to further reduce to some known results. Finally, we present a systematic study of the various fundamental properties of the class of hypergeometric functions introduced here.

Keywords— *Four-Parameter gamma function, Ultra gamma function, Two-parameter gamma function, Two Parameter Pochhammer symbol, Gauss hypergeometric function, Generalized hypergeometric function, Hypergeometric generating functions, Four Parameter Pochhammer symbol, K-Bessel function*

1.INTRODUCTION

The Four Parameter Gamma Function is defined by [14] in the form,

$$_{\delta,a}\Gamma_{\rho,b}(x)=\Gamma(\delta,a;\rho,b)(x)=\int_0^\infty t^{x-1}e^{-\frac{t^\delta}{a}-\frac{t^{-\rho}}{b}}dt. \tag{1}$$

Where, $x\in C/\delta Z^-;\delta,\rho,a,b\in R^+-\{0\}$ and $Re(x-\rho n)>0,n\in N$.

The p - k Gamma Function (i.e. Two Parameter Gamma Function), $_p\Gamma_k(x)$ is given by [16], For $x\in C/kZ^-;k,p\in R^+-\{0\}$ and $Re(x)>0,n\in N$, is:

$$_p\Gamma_k(x)=\frac{1}{k}\lim_{n\to\infty}\frac{n!\,p^{n+1}(np)^{\frac{x}{k}}}{_p(x)_{n+1,k}}. \tag{2}$$

or,

$$_p\Gamma_k(x)=\frac{1}{k}\lim_{n\to\infty}\frac{n!\,p^{n+1}(np)^{\frac{x}{k}-1}}{_p(x)_{n,k}}. \tag{3}$$

And the integral representation of p - k Gamma Function is given by

$$_p\Gamma_k(x)=\int_0^\infty e^{-\frac{t^k}{p}}t^{x-1}dt. \tag{4}$$

The classical Pochhammer symbol $(\lambda)_v$ $(\lambda,v\in C)$ is defined in terms of the gamma function, by

$$(\lambda)_v=\frac{\Gamma(\lambda+v)}{\Gamma(\lambda)}=\begin{cases}1 & (v=0;\lambda\in C\setminus\{0\}),\\ \lambda(\lambda+1)\ldots(\lambda+(n-1)) & (v=n\in N;\lambda\in C);\end{cases} \tag{5}$$

The Bessel function is given by [19],

$$J_v(z)=\sum_{n=0}^\infty\frac{(-1)^n(\frac{z}{2})^{2n+v}}{\Gamma(n+v+1)(n!)}. \tag{6}$$

where $\nu \in I$. If ν is a negative integer, then

$$J_{-\nu}(z) = (-1)^{\nu} J_{\nu}(z). \tag{7}$$

The K-Bessel function is given by [15,17,18]

$$J_{\nu}^{k}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{z}{2})^{2n+(\frac{\nu}{k})}}{\Gamma_k(nk+\nu+k)(n!)}. \tag{8}$$

where $k \in R^{+}$, $\nu \in I$ and $\nu > -k$.
If ν is a negative integer, then

$$J_{-\nu}^{k}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{z}{2})^{2n-(\frac{\nu}{k})}}{\Gamma_k(nk-\nu+k)(n!)}. \tag{9}$$

where $k \in R^{+}$, $\nu \in I$ and $\nu > -k$.
Some Recurrence formulas of Pochhammer symbol of Ultra Gamma Function given by ([14] In Theorem 2.1,2.4 and 3.1.)

If $\lambda \in C \setminus \delta z^{-}$; $\delta, k, \rho, b, c \in R^{+} - \{0\}$, $Re(\lambda - \rho n) > 0$, $n \in N$ and $\frac{\rho}{\delta} \in N$, then

(i) $$\Gamma(\delta, b; -\rho, c)(\lambda + m\delta) = \frac{b^{\frac{\lambda}{\delta}} \Gamma(\frac{\lambda}{\delta})}{\delta} \; {}_{\frac{\rho}{\delta}}F_0[(\frac{\lambda - r\delta - \delta}{\rho})_{r=1,2,\ldots,\frac{\rho}{\delta}}; -; -\frac{1}{c}(\frac{b\rho}{\delta})^{\frac{\rho}{\delta}}]. \tag{10}$$

If $\lambda \in C \setminus \delta z^{-}$; $\delta, k, \rho, b, c \in R^{+} - \{0\}$, $Re(\lambda - \rho n) > 0$, and $n \in N$, then

(ii) $$\lambda \Gamma(\delta, b; \rho, c)(\lambda) = (\frac{\delta}{b} \Gamma(\delta, b; \rho, c)(\lambda + \delta) - \frac{\rho}{c} \Gamma(\delta, b; \rho, c)(\lambda - \rho)). \tag{11}$$

If $\lambda \in C \setminus \delta z^{-}$; $\delta, \rho, b, c \in R^{+} - \{0\}$, $Re(\lambda - \rho n) > 0$, and $n \in N$, then

(iii) $$\Gamma(\delta, b; \rho, c)(\lambda) = (\frac{k}{\delta} \Gamma(k, b; \frac{k\rho}{\delta}, c)(\frac{k\lambda}{\delta}). \tag{12}$$

2. A POCHHAMMER SYMBOL OF ULTRA GAMMA FUNCTION

The Pochhammer symbol of Ultra Gamma function ${}^{\delta,a}(x)_n^{\rho,b}$ is defined by us as,

$${}^{\delta,a}(x)_n^{\rho,b} = \begin{cases} \dfrac{\Gamma(\delta, a; \rho, b)(x+n)}{\Gamma(x)}; & \left(x \in C \setminus \delta Z^{-}; \delta, \rho, a, b \in R^{+} - \{0\}\right) \\ & \left(Re(x - \rho n) > 0, n \in N\right), \\ \dfrac{e^{-\frac{1}{b}} \Gamma(x+n)}{\Gamma(x)} = e^{-\frac{1}{b}} (x)_n; & \left(\delta = a = 1, \rho = 0; n \in N\right) \end{cases} \tag{13}$$

Theorem 1: Let, $m, n \in N_0$, $\delta, \rho, b, c \in R^{+} - \{0\}$. Then

$${}^{\delta,b}(\lambda)_{n+m}^{\rho,c} = (\lambda)_n \; {}^{\delta,b}(\lambda+n)_m^{\rho,c}. \tag{14}$$

Proof: From definition (13),

$${}^{\delta,b}(\lambda)_{m+n}^{\rho,c} = \frac{\Gamma(\delta, b; \rho, c)(\lambda+n+m)}{\Gamma(\lambda)} = \frac{\Gamma(\delta, b; \rho, c)(\lambda+n+m) \quad \Gamma(\lambda+n)}{\Gamma(\lambda) \quad \Gamma(\lambda+n)}.$$

and using (5), we have

$${}^{\delta,b}(\lambda)_{m+n}^{\rho,c} = (\lambda)_n \; {}^{\delta,b}(\lambda+n)_m^{\rho,c}.$$

Corollary 1: Let, $\lambda \in C \setminus \delta Z^{-}$, $\delta, \rho, b, c \in R^{+} - \{0\}$ and $T \in N$. Then,

$${}^{\delta,b}(\lambda)_{n+m+l}^{\rho,c} = (\lambda)_n \quad (\lambda+n)_m \quad {}^{\delta,b}(\lambda+m+n)_l^{\rho,c}. \tag{15}$$

$${}^{\delta,b}(\lambda)_{n-m+l}^{\rho,c} = \frac{(-1)^m (\lambda)_n}{(1-\lambda-n)_m} \; {}^{\delta,b}(\lambda+n-m)_l^{\rho,c}. \tag{16}$$

$${}^{\delta,b}(\lambda)_{2m+l}^{\rho,c} = 2^{2m} \quad (\frac{\lambda}{2})_m \quad (\frac{\lambda+1}{2})_m \quad {}^{\delta,b}(\lambda+2m)_l^{\rho,c}. \tag{17}$$

$$^{\delta,b}(\lambda)^{\rho,c}_{Tm+l} = T^{Tm} \quad (\frac{\lambda}{T})_m \quad (\frac{\lambda+1}{T})_m \ldots (\frac{\lambda+T-1}{T})_m \quad {}^{\delta,b}(\lambda+Tm)^{\rho,c}_l. \tag{18}$$

$$^{\delta,b}(\lambda+n)^{\rho,c}_{n+l} = (\lambda+n)_n \quad {}^{\delta,b}(\lambda+2n)^{\rho,c}_l = \frac{(\lambda)_{2n}}{(\lambda)_n} \quad {}^{\delta,b}(\lambda+2n)^{\rho,c}_l. \tag{19}$$

$$^{\delta,b}(\lambda+m)^{\rho,c}_{n+l} = \frac{(\lambda)_n(\lambda+n)_m}{(\lambda)_m} \quad {}^{\delta,b}(\lambda+m+n)^{\rho,c}_l. \tag{20}$$

$$^{\delta,b}(\lambda+km)^{\rho,c}_{kn+l} = \frac{(\lambda)_{kn+km}}{(\lambda)_{km}} \quad {}^{\delta,b}(\lambda+km+kn)^{\rho,c}_l. \tag{21}$$

$$^{\delta,b}(\lambda-n)^{\rho,c}_{n+l} = (-1)^n \quad (1-\lambda)_n \quad {}^{\delta,b}(\lambda)^{\rho,c}_l. \tag{22}$$

$$^{\delta,b}(\lambda-m)^{\rho,c}_{n+l} = \frac{(1-\lambda)_m(\lambda)_n}{(1-\lambda-n)_m} \quad {}^{\delta,b}(\lambda+n-m)^{\rho,c}_l. \tag{23}$$

$$^{\delta,b}(\lambda-km)^{\rho,c}_{kn+l} = (-1)^{km}(\lambda)_{kn-km}(1-\lambda)_{km} \quad {}^{\delta,b}(\lambda+km+kn)^{\rho,c}_l. \tag{24}$$

$$^{\delta,b}(\lambda+m)^{\rho,c}_{n-m+l} = \frac{(\lambda)_n}{(\lambda)_m} \quad {}^{\delta,b}(\lambda+n)^{\rho,c}_l. \tag{25}$$

$$^{\delta,b}(\lambda-m)^{\rho,c}_{n-m+l} = \frac{(-1)^m(\lambda)_n(1-\lambda)_m}{(1-\lambda-n)_{2m}} \quad {}^{\delta,b}(\lambda+n-2m)^{\rho,c}_l. \tag{26}$$

$$^{\delta,b}(-\lambda)^{\rho,c}_{n+l} = (-1)^n(\lambda-n+1)_n \quad {}^{\delta,b}(-\lambda+n)^{\rho,c}_l. \tag{27}$$

Some Recurrence formulas of Pochhammer symbol of Ultra Gamma Function

If $\lambda \in C \setminus \delta z^-$; $\delta,\rho,b,c \in R^+ - \{0\}$, $Re(\lambda-\rho n) > 0$, $n \in N$ and $\frac{\rho}{\delta} \in N$, then Using ([14] Theorem 3.1) we get,

(i)
$$^{\delta,b}(\lambda)^{-\rho,c}_{m\delta} = \frac{\Gamma(\delta,b;-\rho,c)(\lambda+m\delta)}{\Gamma(\lambda)}. \tag{28}$$

$$^{\delta,b}(\lambda)^{-\rho,c}_{m\delta} = \frac{b^{\frac{\lambda}{\delta}}\Gamma(\frac{\lambda}{\delta})}{\delta\Gamma(\lambda)} \quad {}_{\frac{\rho}{\delta}}F_0[(\frac{\lambda-r\delta-\delta}{\rho})_{r=1,2,\ldots,\frac{\rho}{\delta}};-;-\frac{1}{c}(\frac{b\rho}{\delta})^{\frac{\rho}{\delta}}]. \tag{29}$$

If $\lambda \in C \setminus \delta z^-$; $\delta,\rho,b,c \in R^+ - \{0\}$, $Re(\lambda-\rho n) > 0$, and $n \in N$, then

(ii)
$$\lambda \quad {}^{\delta,b}(\lambda)^{\rho,c}_n = \frac{1}{\Gamma(\lambda)}(\frac{\delta}{b}\Gamma(\delta,b;\rho,c)(\lambda+\delta) - \frac{\rho}{c}\Gamma(\delta,b;\rho,c)(\lambda-\rho)). \tag{30}$$

If $\lambda \in C \setminus \delta z^-$; $\delta,k,\rho,b,c \in R^+ - \{0\}$, $Re(\lambda-\rho n) > 0$, and $n \in N$, then

(iii)
$$^{\delta,b}(\lambda)^{\rho,c} = \frac{1}{\Gamma(\lambda)}(\frac{k}{\delta}\Gamma(k,b;\frac{k\rho}{\delta},c)(\frac{k\lambda}{\delta}). \tag{31}$$

3. EXTENSION AND GENERALIZATION OF THE HYPERGEOMETRIC FUNCTION

In terms of Ultra Gamma function, an extension of the generalized hypergeometric function ${}_rF_s$ of r numerator parameters $a_1,\ldots,a_r$ and s denominator parameters $d_1,\ldots,d_s$ can now be given as follows:

$${}_rF_s\left[\begin{matrix} ^{\delta_1,b_1}(a_1)^{\rho_1,c_1}, {}^{\delta_2,b_2}(a_2)^{\rho_2,c_2},\ldots,{}^{\delta_r,b_r}(a_r)^{\rho_r,c_r}; \\ z \\ ^{\alpha_1,f_1}(d_1)^{\eta_1,g_1}, {}^{\alpha_2,f_2}(d_2)^{\eta_2,g_2},\ldots,{}^{\alpha_s,f_s}(d_s)^{\eta_s,g_s}; \end{matrix}\right] = \sum_{n=0}^{\infty}\frac{^{\delta_1,b_1}(a_1)^{\rho_1,c_1}_n, {}^{\delta_2,b_2}(a_2)^{\rho_2,c_2}_n \ldots, {}^{\delta_r,b_r}(a_r)^{\rho_r,c_r}_n}{^{\alpha_1,f_1}(d_1)^{\eta_1,g_1}_n,\ldots, {}^{\alpha_s,f_s}(d_s)^{\eta_s,g_s}_n}\frac{z^n}{n!}. \tag{32}$$

provided that the series on the right hand side converges,

$a_r \in C$, $\delta_r, b_r, c_r, \rho_r, \alpha_s, f_s, g_s, \eta_s \in R^+ - \{0\}$ where $r = 1,2,\ldots,r$ and $d_s \in C \setminus Z_0^-$ where $s = 1,2,\ldots,s; Z_0^- = 0,-1,-2,\ldots$

In particular, the corresponding extensions of the confluent hypergeometric function ${}_1F_1$ and the Gauss hypergeometric function ${}_2F_1$ are given by

$${}_1F_1[^{\delta_1,b_1}(a_1)^{\rho_1,c_1};{}^{\delta_2,b_2}(a_2)^{\rho_2,c_2};z] = \sum_{n=0}^{\infty}\frac{^{\delta_1,b_1}(a_1)^{\rho_1,c_1}_n}{^{\delta_2,b_2}(a_2)^{\rho_2,c_2}_n}\frac{z^n}{n!}. \tag{33}$$

and,

$$ {}_2F_1[{}^{\delta_1,b_1}(a_1)^{\rho_1,c_1},{}^{\delta_2,b_2}(a_2)^{\rho_2,c_2};{}^{\delta_3,b_3}(a_3)^{\rho_3,c_3};z]=\sum_{n=0}^{\infty}\frac{{}^{\delta_1,b_1}(a_1)_n^{\rho_1,c_1}\;{}^{\delta_2,b_2}(a_2)_n^{\rho_2,c_2}}{{}^{\delta_3,b_3}(a_3)_n^{\rho_3,c_3}}\frac{z^n}{n!}. \tag{34}$$

respectively.

If we put $a_1 = a_3$, $\delta_1 = \delta_3$, $b_1 = b_3$, $\rho_1 = \rho_3$, $c_1 = c_3$ and $\delta_2 = b_2 = 1$, $\rho_2 = 0$ then,

$$ {}_2F_1[{}^{\delta_1,b_1}(a_1)^{\rho_1,c_1},{}^{\delta_2,b_2}(a_2)^{\rho_2,c_2};{}^{\delta_1,b_1}(a_1)^{\rho_1,c_1};z]=e^{-\frac{1}{c_2}}\sum_{n=0}^{\infty}\frac{(a_2)_n z^n}{n!}=e^{-\frac{1}{c_2}}\quad(1-z)^{-a_2}. \tag{35}$$

Theorem 2: The integral representation is given by,

$$ {}_rF_s\left[\begin{matrix}{}^{\delta_1,b_1}(a_1)^{\rho_1,c_1},{}^{\delta_2,b_2}(a_2)^{\rho_2,c_2},\ldots,{}^{\delta_r,b_r}(a_r)^{\rho_r,c_r}; \\ & z \\ {}^{\alpha_1,f_1}(d_1)^{\eta_1,g_1},{}^{\alpha_2,f_2}(d_2)^{\eta_2,g_2},\ldots,{}^{\alpha_s,f_s}(d_s)^{\eta_s,g_s};\end{matrix}\right]=\frac{1}{\Gamma(a_1)}\int_0^{\infty}t^{a_1-1}e^{-\frac{t^{\delta_1}}{b_1}-\frac{t^{-\rho_1}}{c_1}}\times$$

$$ {}_{r-1}F_s\left[\begin{matrix}{}^{\delta_2,b_2}(a_2)^{\rho_2,c_2},\ldots,{}^{\delta_r,b_r}(a_r)^{\rho_r,c_r}; \\ & zt \\ {}^{\alpha_1,f_1}(d_1)^{\eta_1,g_1},{}^{\alpha_2,f_2}(d_2)^{\eta_2,g_2},\ldots,{}^{\alpha_s,f_s}(d_s)^{\eta_s,g_s};\end{matrix}\right]dt. \tag{36}$$

Proof: Using definition (13) and (1), we immedeted get desired result (36).

Theorem 3: The integral representation is given by,

$$ {}_rF_s\left[\begin{matrix}{}^{\delta_1,b_1}(a_1)^{\rho_1,c_1},{}^{\delta_2,b_2}(a_2)^{\rho_2,c_2},\ldots,{}^{\delta_r,b_r}(a_r)^{\rho_r,c_r}; \\ & z \\ {}^{\alpha_1,f_1}(d_1)^{\eta_1,g_1},{}^{\alpha_2,f_2}(d_2)^{\eta_2,g_2},\ldots,{}^{\alpha_s,f_s}(d_s)^{\eta_s,g_s};\end{matrix}\right]=\frac{(b_1)^{\frac{a_1}{\delta_1}}\Gamma(d_1)\alpha_1}{\Gamma(a_1)\delta_1(f_1)^{\frac{d_1}{\alpha_1}}}\quad\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^m(b_1)^{\frac{n-\rho_1 m}{\delta_1}}\;l!(g_1)^l}{(-1)^l(f_1)^{\frac{n-\eta_1 l}{\alpha_1}}\;m!\quad(c_1^m)}\times$$

$$\frac{\Gamma(\frac{2(a_1)+n-(\rho_1)m}{(\delta_1)})\quad\Gamma(\frac{d_1}{\alpha_1})\,{}^{\delta_2,b_2}(a_2)_n^{\rho_2,c_2}\quad\ldots{}^{\delta_r,b_r}(a_r)_n^{\rho_r,c_r}}{\Gamma(\frac{a_1}{\delta_1})\quad B(\frac{d_1-\eta_1 l+n}{\alpha_1},\frac{d_1}{\alpha_1})\quad\Gamma(\frac{2d_1+n-\eta_1 l}{\alpha_1})\,{}^{\alpha_2,f_2}(d_2)_n^{\eta_2,g_2}\quad\ldots{}^{\alpha_s,f_s}(d_s)_n^{\eta_s,g_s}}\frac{z^n}{n!}\times\int_0^1 t^{\frac{a_1-\rho_1 m}{\delta_1}}\quad(1-t)^{\frac{a_1}{\delta_1}}\quad dt. \tag{37}$$

Proof: By using definition of ultra gamma function and hypergeometric function

$$ {}_rF_s\left[\begin{matrix}{}^{\delta_1,b_1}(a_1)^{\rho_1,c_1},{}^{\delta_2,b_2}(a_2)^{\rho_2,c_2},\ldots,{}^{\delta_r,b_r}(a_r)^{\rho_r,c_r}; \\ & z \\ {}^{\alpha_1,f_1}(d_1)^{\eta_1,g_1},{}^{\alpha_2,f_2}(d_2)^{\eta_2,g_2},\ldots,{}^{\alpha_s,f_s}(d_s)^{\eta_s,g_s};\end{matrix}\right]=\sum_{n=0}^{\infty}\frac{{}^{\delta_1,b_1}(a_1)_n^{\rho_1,c_1}\quad{}^{\delta_2,b_2}(a_2)_n^{\rho_2,c_2}\quad\ldots{}^{\delta_r,b_r}(a_r)_n^{\rho_r,c_r}}{{}^{\alpha_1,f_1}(d_1)_n^{\eta_1,g_1}\quad{}^{\alpha_2,f_2}(d_2)_n^{\eta_2,g_2}\quad\ldots{}^{\alpha_s,f_s}(d_s)_n^{\eta_s,g_s}}\frac{z^n}{n!}$$

$$=\sum_{n=0}^{\infty}\frac{\Gamma(d_1)\quad\Gamma_{\delta_1,b_1;\rho_1,c_1}(a_1+n)\quad{}^{\delta_2,b_2}(a_2)_n^{\rho_2,c_2}\quad\ldots{}^{\delta_r,b_r}(a_r)_n^{\rho_r,c_r}}{\Gamma(a_1)\quad\Gamma_{\alpha_1,f_1;\eta_1,g_1}(d_1+n)\quad{}^{\alpha_2,f_2}(d_2)_n^{\eta_2,g_2}\quad\ldots{}^{\alpha_s,f_s}(d_s)_n^{\eta_s,g_s}}\frac{z^n}{n!}$$

$$=\frac{\Gamma(d_1)}{\Gamma(a_1)}\times\quad\sum_{n=0}^{\infty}\frac{\sum_{m=0}^{\infty}\frac{(-1)^m(b_1)^{\frac{(a_1)+n-(\rho_1)m}{(\delta_1)}}}{m!(c_1)^m\delta_1}\Gamma(\frac{(a_1)+n-(\rho_1)m}{(\delta_1)})\quad{}^{\delta_2,b_2}(a_2)_n^{\rho_2,c_2}\ldots{}^{\delta_r,b_r}(a_r)_n^{\rho_r,c_r}}{\sum_{l=0}^{\infty}\frac{(-1)^l(f_1)^{\frac{d_1+n-\eta_1 l}{\alpha_1}}}{l!(g_1)^l\alpha_1}\Gamma(\frac{d_1+n-\eta_1 l}{\alpha_1})\quad{}^{\alpha_2,f_2}(d_2)_n^{\eta_2,g_2}\ldots{}^{\alpha_s,f_s}(d_s)_n^{\eta_s,g_s}}\frac{z^n}{n!}$$

$$=\frac{(b_1)^{\frac{a_1}{\delta_1}}\quad\Gamma(d_1)\alpha_1}{\Gamma(a_1)\delta_1(f_1)^{\frac{d_1}{\alpha_1}}}\times\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^m(b_1)^{\frac{n-\rho_1 m}{\delta_1}}\quad l!(g_1)^m}{m!\quad(c_1^m)(-1)^l(f_1)^{\frac{n-\eta_1 l}{\alpha_1}}}\times\frac{\Gamma(\frac{(a_1)+n-(\rho_1)m}{(\delta_1)})\quad{}^{\delta_2,b_2}(a_2)_n^{\rho_2,c_2}\quad\ldots{}^{\delta_r,b_r}(a_r)_n^{\rho_r,c_r}}{\Gamma(\frac{d_1+n-\eta_1 l}{\alpha_1})\quad{}^{\alpha_2,f_2}(d_2)_n^{\eta_2,g_2}\quad\ldots{}^{\alpha_s,f_s}(d_s)_n^{\eta_s,g_s}}\frac{z^n}{n!}$$

$$= \frac{(b_1)^{\frac{a_1}{\delta_1}}\Gamma(d_1)\alpha_1}{\Gamma(a_1)\delta_1(f_1)^{\frac{d_1}{\alpha_1}}}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^m(b_1)^{\frac{n-\rho_1 m}{\delta_1}}\, l!(g_1)^l}{m!\;(c_1^m)(-1)^l(f_1)^{\frac{n-\eta_1 l}{\alpha_1}}}\times$$

$$\frac{\Gamma(\frac{(a_1)+n-(\rho_1)m}{(\delta_1)})\;\Gamma(\frac{a_1}{\delta_1})\;\Gamma(\frac{2a_1-\rho_1 m+n}{\delta_1})\;\Gamma(\frac{d_1}{\alpha_1})\;\Gamma(\frac{2d_1-\eta_1 l+n}{\alpha_1})}{\Gamma(\frac{d_1+n-\eta_1 l}{\alpha_1})\;\Gamma(\frac{d_1}{\alpha_1})\;\Gamma(\frac{2d_1-\eta_1 l+n}{\alpha_1})\;\Gamma(\frac{a_1}{\delta_1})\;\Gamma(\frac{2a_1-\rho_1 m+n}{\delta_1})}\times\frac{{}^{\delta_2,b_2}(a_2)_n^{\rho_2,c_2}\;\ldots{}^{\delta_r,b_r}(a_r)_n^{\rho_r,c_r}}{{}^{\alpha_2,f_2}(d_2)_n^{\eta_2,g_2}\;\ldots{}^{\alpha_s,f_s}(d_s)_n^{\eta_s,g_s}}\frac{z^n}{n!}$$

$$= \frac{(b_1)^{\frac{a_1}{\delta_1}}\Gamma(d_1)\alpha_1}{\Gamma(a_1)\delta_1(f_1)^{\frac{d_1}{\alpha_1}}}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^m(b_1)^{\frac{n-\rho_1 m}{\delta_1}}\, l!(g_1)^l}{(-1)^l(f_1)^{\frac{n-\eta_1 l}{\alpha_1}}\;m!\;(c_1^m)}\times$$

$$\frac{B(\frac{a_1-\rho_1 m+n}{\delta_1},\frac{a_1}{\delta_1})\Gamma(\frac{2(a_1)+n-(\rho_1)m}{(\delta_1)})\Gamma(\frac{d_1}{\alpha_1})\;{}^{\delta_2,b_2}(a_2)_n^{\rho_2,c_2}\;\ldots{}^{\delta_r,b_r}(a_r)_n^{\rho_r,c_r}}{B(\frac{d_1-\eta_1 l+n}{\alpha_1},\frac{d_1}{\alpha_1})\;\Gamma(\frac{2d_1+n-\eta_1 l}{\alpha_1})\;\Gamma(\frac{a_1}{\delta_1})\;{}^{\alpha_2,f_2}(d_2)_n^{\eta_2,g_2}\;\ldots{}^{\alpha_s,f_s}(d_s)_n^{\eta_s,g_s}}\frac{z^n}{n!}$$

$$= \frac{(b_1)^{\frac{a_1}{\delta_1}}\Gamma(d_1)\alpha_1}{\Gamma(a_1)\delta_1(f_1)^{\frac{d_1}{\alpha_1}}}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^m(b_1)^{\frac{n-\rho_1 m}{\delta_1}}\, l!(g_1)^l}{(-1)^l(f_1)^{\frac{n-\eta_1 l}{\alpha_1}}\;m!\;(c_1^m)}\times$$

$$\frac{\int_o^1 t^{\frac{a_1-\rho_1 m}{\delta_1}}(1-t)^{\frac{a_1}{\delta_1}}dt\;\;\Gamma(\frac{2(a_1)+n-(\rho_1)m}{(\delta_1)})\;\;\Gamma(\frac{d_1}{\alpha_1})^{\delta_2,b_2}(a_2)_n^{\rho_2,c_2}\;\ldots{}^{\delta_r,b_r}(a_r)_n^{\rho_r,c_r}}{\Gamma(\frac{a_1}{\delta_1})\;\;B(\frac{d_1-\eta_1 l+n}{\alpha_1},\frac{d_1}{\alpha_1})\;\;\Gamma(\frac{2d_1+n-\eta_1 l}{\alpha_1})^{\alpha_2,f_2}(d_2)_n^{\eta_2,g_2}\;\ldots{}^{\alpha_s,f_s}(d_s)_n^{\eta_s,g_s}}\frac{z^n}{n!}$$

$$= \frac{(b_1)^{\frac{a_1}{\delta_1}}\Gamma(d_1)\alpha_1}{\Gamma(a_1)\delta_1(f_1)^{\frac{d_1}{\alpha_1}}}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^m(b_1)^{\frac{n-\rho_1 m}{\delta_1}}\, l!(g_1)^l}{(-1)^l(f_1)^{\frac{n-\eta_1 l}{\alpha_1}}\;m!\;(c_1^m)}\times$$

$$\frac{\Gamma(\frac{2(a_1)+n-(\rho_1)m}{(\delta_1)})\;\;\Gamma(\frac{d_1}{\alpha_1})^{\delta_2,b_2}(a_2)_n^{\rho_2,c_2}\;\ldots{}^{\delta_r,b_r}(a_r)_n^{\rho_r,c_r}}{\Gamma(\frac{a_1}{\delta_1})\;\;B(\frac{d_1-\eta_1 l+n}{\alpha_1},\frac{d_1}{\alpha_1})\;\;\Gamma(\frac{2d_1+n-\eta_1 l}{\alpha_1})^{\alpha_2,f_2}(d_2)_n^{\eta_2,g_2}\;\ldots{}^{\alpha_s,f_s}(d_s)_n^{\eta_s,g_s}}\frac{z^n}{n!}\times\int_0^1 t^{\frac{a_1-\rho_1 m}{\delta_1}}(1-t)^{\frac{a_1}{\delta_1}}dt$$

Hence Proved.

Theorem 4: The following derivative formula holds true:

$$\frac{d^m}{dz^m}\left\{{}_rF_s\left[\begin{matrix}{}^{\delta_1,b_1}(a_1)^{\rho_1,c_1},{}^{\delta_2,b_2}(a_2)^{\rho_2,c_2},\ldots,{}^{\delta_r,b_r}(a_r)^{\rho_r,c_r};\\ z\\ {}^{\alpha_1,f_1}(d_1)^{\eta_1,g_1},{}^{\alpha_2,f_2}(d_2)^{\eta_2,g_2},\ldots,{}^{\alpha_s,f_s}(d_s)^{\eta_s,g_s};\end{matrix}\right]\right\}=\frac{(a_1)_m,(a_2)_m,\ldots,(a_r)_m}{(b_1)_m,(b_2)_m,\ldots,(b_s)_m}\times \tag{38}$$

$${}_rF_s\left[\begin{matrix}{}^{\delta_1,b_1}(a_1+m)^{\rho_1,c_1},{}^{\delta_2,b_2}(a_2+m)^{\rho_2,c_2},\ldots,{}^{\delta_r,b_r}(a_r+m)^{\rho_r,c_r};\\ z\\ {}^{\alpha_1,f_1}(d_1+m)^{\eta_1,g_1},{}^{\alpha_2,f_2}(d_2+m)^{\eta_2,g_2},\ldots,{}^{\alpha_s,f_s}(d_s+m)^{\eta_s,g_s};\end{matrix}\right].\text{for }\;(m\in N_0).$$

Proof: This result is obviously valid in the trivial case when $m=0$. For $m=1$, by using the series representation (32) of ${}_rF_s$,

$$\frac{d}{dz}\sum_{n=0}^{\infty}\frac{{}^{\delta_1,b_1}(a_1)_n^{\rho_1,c_1}\;{}^{\delta_2,b_2}(a_2)_n^{\rho_2,c_2}\ldots{}^{\delta_r,b_r}(a_r)_n^{\rho_r,c_r}}{{}^{\alpha_1,f_1}(d_1)_n^{\eta_1,g_1}\;{}^{\alpha_2,f_2}(d_2)_n^{\eta_2,g_2}\ldots{}^{\alpha_s,f_s}(d_s)_n^{\eta_s,g_s}}\frac{z^n}{n!}$$

$$=\sum_{n=1}^{\infty}\frac{{}^{\delta_1,b_1}(a_1)_n^{\rho_1,c_1}\;{}^{\delta_2,b_2}(a_2)_n^{\rho_2,c_2}\ldots{}^{\delta_r,b_r}(a_r)_n^{\rho_r,c_r}}{{}^{\alpha_1,f_1}(d_1)_n^{\eta_1,g_1}\;{}^{\alpha_2,f_2}(d_2)_n^{\eta_2,g_2}\ldots{}^{\alpha_s,f_s}(d_s)_n^{\eta_s,g_s}}\frac{z^{n-1}}{(n-1)!},=\sum_{n=0}^{\infty}\frac{{}^{\delta_1,b_1}(a_1)_{n+1}^{\rho_1,c_1}\;{}^{\delta_2,b_2}(a_2)_{n+1}^{\rho_2,c_2}\ldots{}^{\delta_r,b_r}(a_r)_{n+1}^{\rho_r,c_r}}{{}^{\alpha_1,f_1}(d_1)_{n+1}^{\eta_1,g_1}\;{}^{\alpha_2,f_2}(d_2)_{n+1}^{\eta_2,g_2}\ldots{}^{\alpha_s,f_s}(d_s)_{n+1}^{\eta_s,g_s}}\frac{z^n}{(n)!}.$$

which, in view of (14) yeilds

$$\frac{d}{dz}\{{}_rF_s\left[\begin{matrix}{}^{\delta_1,b_1}(a_1)^{\rho_1,c_1},\;{}^{\delta_2,b_2}(a_2)^{\rho_2,c_2}\ldots,{}^{\delta_r,b_r}(a_r)^{\rho_r,c_r};\\ & z\\ {}^{\alpha_1,f_1}(d_1)^{\eta_1,g_1},{}^{\alpha_2,f_2}(d_2)^{\eta_2,g_2},\ldots,{}^{\alpha_s,f_s}(d_s)^{\eta_s,g_s};\end{matrix}\right]\}$$

$$=\frac{(a_1)(a_2)\ldots(a_r)}{(b_1)(b_2)\ldots(b_s)}\times{}_rF_s\left[\begin{matrix}{}^{\delta_1,b_1}(a_1+1)^{\rho_1,c_1},{}^{\delta_2,b_2}(a_2+1)^{\rho_2,c_2},\ldots,{}^{\delta_r,b_r}(a_r+1)^{\rho_r,c_r};\\ & z\\ {}^{\alpha_1,f_1}(d_1+1)^{\eta_1,g_1},{}^{\alpha_2,f_2}(d_2+1)^{\eta_2,g_2},\ldots,{}^{\alpha_s,f_s}(d_s+1)^{\eta_s,g_s};\end{matrix}\right].$$

The general result (38) can now be easily derived by using the principle of mathematical induction on $m\in N_0$.

We state the following results without proof. Each of these result would follow readily from the corresponding known result involving the generalized hypergeometric functions, which are asserted by Theorems 2.

Corollary 2: Each of the following integral representations holds true:

$${}_1F_1[{}^{\delta_1,b_1}(a_1)^{\rho_1,c_1};{}^{\delta_2,b_2}(a_2)^{\rho_2,c_2};z]=\frac{1}{\Gamma(a_1)}\int_0^{\infty}t^{a_1-1}\exp(-\frac{t^{\delta_1}}{b_1}-\frac{t^{-\rho_1}}{c_1})\,{}_0F_1[\_;{}^{\delta_2,b_2}(a_2)^{\rho_2,c_2};zt]dt. \quad (39)$$

and

$${}_2F_1[{}^{\delta_1,b_1}(a_1)^{\rho_1,c_1},{}^{\delta_2,b_2}(a_2)^{\rho_2,c_2};{}^{\delta_3,b_3}(a_3)^{\rho_3,c_3};z]$$
$$=\frac{1}{\Gamma(a_1)}\int_0^{\infty}t^{a_1-1}\exp(-\frac{t^{\delta_1}}{b_1}-\frac{t^{-\rho_1}}{c_1})\,{}_1F_1[{}^{\delta_2,b_2}(a_2)^{\rho_2,c_2};{}^{\delta_3,b_3}(a_3)^{\rho_3,c_3};zt]dt. \quad (40)$$

Provided that the integrals involved are convergent.

Theorem 5: Each of the following integral representations holds true:

$${}_1F_1[{}^{\delta,b}(a)^{\rho,c};\frac{\mu}{k}+1;-z]=\frac{\Gamma_k(\mu+k)}{\Gamma(a)}(zk)^{-\frac{\mu}{2k}}\int_0^{\infty}t^{a-\frac{\mu}{2k}-1}e^{-\frac{t^{\delta}}{b}-\frac{t^{-\rho}}{c}}J_{\mu}^{k}(2(ztk)^{\frac{1}{2}})dt. \quad (41)$$

and,

$${}_1F_1[{}^{\delta,b}(a)^{\rho,c};\frac{-\mu}{k}+1;-z]=\frac{\Gamma_k(-\mu+k)}{\Gamma(a)}(zk)^{\frac{-\mu}{2k}}\int_0^{\infty}t^{a+\frac{\mu}{2k}-1}e^{-\frac{t^{\delta}}{b}-\frac{t^{-\rho}}{c}}J_{-\mu}^{k}(2(ztk)^{\frac{1}{2}})dt. \quad (42)$$

Proof: The Hypergeometric function ${}_1F_1$ is given as,

$${}_1F_1[{}^{\delta,b}(a)^{\rho,c};\frac{\mu}{k}+1;-z]=\sum_{n=0}^{\infty}\frac{{}^{\delta,b}(a)_n^{\rho,c}}{(\frac{\mu}{k}+1)_n}\frac{(-z)^n}{n!}$$

$$=\frac{1}{\Gamma(a)}\sum_{n=0}^{\infty}\frac{\int_0^{\infty}t^{a+n-1}e^{-\frac{t^{\delta}}{b}-\frac{t^{-\rho}}{c}}}{(\frac{\mu}{k}+1)_n}\frac{(-z)^n}{n!}dt$$

$$=\frac{\Gamma(\frac{\mu+k}{k})}{\Gamma a}\int_0^\infty t^{a-\frac{\mu}{2k}-1} \quad e^{-\frac{t^\delta}{b}-\frac{t^{-\rho}}{c}}(z)^{-\frac{\mu}{2k}}\sum_{n=0}^\infty \frac{(-1)^n (zt)^{n+\frac{\mu}{2k}}}{\Gamma(\frac{\mu+k+nk}{k})n!}dt$$

$$=\frac{\Gamma_k(\mu+k)}{\Gamma a}\int_0^\infty t^{a-\frac{\mu}{2k}-1} e^{-\frac{t^\delta}{b}-\frac{t^{-\rho}}{c}}(zk)^{-\frac{\mu}{2k}}\sum_{n=0}^\infty \frac{(-1)^n (ztk)^{n+\frac{\mu}{2k}}}{\Gamma_k(\mu+k+nk)n!}dt,$$

$${}_1F_1[^{\delta,b}(a)^{\rho,c};\frac{\mu}{k}+1;-z]=\frac{\Gamma_k(\mu+k)}{\Gamma(a)}(zk)^{-\frac{\mu}{2k}}\int_0^\infty t^{a-\frac{\mu}{2k}-1} e^{-\frac{t^\delta}{b}-\frac{t^{-\rho}}{c}} J_\mu^k(2(ztk)^{\frac{1}{2}})dt.$$

Hence Proved.

SPECIAL CASES

Case I: If we put k=1 in equation (38) and (39) respectively, then we get known result of ([11] remark 2 equation 27, page number 488).

$${}_1F_1[^{\delta,b}(a)^{\rho,c};v+1;-z]=\frac{\Gamma(v+1)}{\Gamma(a)}z^{-\frac{v}{2}}\int_0^\infty t^{a-\frac{v}{2}-1} e^{-\frac{t^\delta}{b}-\frac{t^{-\rho}}{c}} \quad J_v(2(zt)^{\frac{1}{2}})dt. \tag{43}$$

and

$${}_1F_1[^{\delta,b}(a)^{\rho,c};-v+1;-z]=\frac{\Gamma(-v+1)}{\Gamma(a)}z^{\frac{v}{2}}\int_0^\infty t^{a+\frac{v}{2}-1} e^{-\frac{t^\delta}{b}-\frac{t^{-\rho}}{c}} \quad J_{-v}(2(zt)^{\frac{1}{2}})dt. \tag{44}$$

This is new result.

4. FAMILIES OF GENERALIZED HYPERGEOMETRIC GENERATING FUNCTIONS

In order to derive several families of generaized hypergeometric generating functions, we find it to be convenient to abbreviate by $\Delta(n,\lambda)$ the following array of T parameters: $\frac{\lambda}{T}, \frac{\lambda+1}{T}, \ldots, \frac{\lambda+T-1}{N} (\lambda \in C; T \in N),$ the array $\Delta(T;\lambda)$ being assumed to be empty when $T=0$.

Theorem 6: The following generating function holds true:

$$\sum_{n=0}^\infty \frac{(\lambda)_n}{n!}\times {}_{r+T}F_s\left[\begin{matrix}\Delta(T;\lambda+n),^{\delta_1,b_1}(a_1)^{\rho_1,c_1},^{\delta_2,b_2}(a_2)^{\rho_2,c_2},\ldots,^{\delta_r,b_r}(a_r)^{\rho_r,c_r}; \\ Z \\ ^{\alpha_1,f_1}(d_1)^{\eta_1,g_1}, \quad ^{\alpha_2,f_2}(d_2)^{\eta_2,g_2},\ldots,^{\alpha_s,f_s}(d_s)^{\eta_s,g_s};\end{matrix}\right]$$

$$=(1-t)^{-\lambda}\times_{r+T}F_s\left[\begin{matrix}\Delta(T;\lambda)^{\delta_1,b_1}(a_1)^{\rho_1,c_1}, \quad ^{\delta_2,b_2}(a_2)^{\rho_2,c_2},\ldots, \quad ^{\delta_r,b_r}(a_r)^{\rho_r,c_r}; \\ \frac{Z}{(1-t)^T} \\ ^{\alpha_1,f_1}(d_1)^{\eta_1,g_1}, \quad ^{\alpha_2,f_2}(d_2)^{\eta_2,g_2},\ldots, \quad ^{\alpha_s,f_s}(d_s)^{\eta_s,g_s};\end{matrix}\right]. \tag{45}$$

$$(|t|<1;\lambda\in C;T\in N),$$

Proof: The derivation of the generating function (45) is based upon the definition (32) and the following elementary identity:

$$\sum_{n=0}^\infty \frac{(\lambda)_n}{n!}z^n=(1-z)^{-\lambda} \qquad (|z|<1;\lambda\in C). \tag{46}$$

The details involved are being omitted here.

Remark 3: Whenever any of the numerator parameters $a_2, a_3, \ldots, a_r$ is a nonpositive integer the series in the definition (29) would terminate and define a generalized hypergeometric polynomial. Theorem 7 below provides a general family of generating functions for such classes of hypergeometric polynomials.

Theorem 7: Each of the following generating functions holds true:

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\times_{r+T}F_s\left[\begin{matrix}\Delta(T;-n), & {}^{\delta_1,b_1}(a_1)^{\rho_1,c_1}, & {}^{\delta_2,b_2}(a_2)^{\rho_2,c_2},\ldots, & {}^{\delta_r,b_r}(a_r)^{\rho_r,c_r}; \\ & & & Z \\ {}^{\alpha_1,f_1}(d_1)^{\eta_1,g_1}, & {}^{\alpha_2,f_2}(d_2)^{\eta_2,g_2},\ldots, & {}^{\alpha_s,f_s}(d_s)^{\eta_s,g_s}; & \end{matrix}\right]t^n$$

$$=(1-t)^{-\lambda}{}_{r+T}F_s\left[\begin{matrix}\Delta(T;\lambda) & {}^{\delta_1,b_1}(a_1)^{\rho_1,c_1}, & {}^{\delta_2,b_2}(a_2)^{\rho_2,c_2},\ldots, & {}^{\delta_r,b_r}(a_r)^{\rho_r,c_r}; \\ & & & \dfrac{-t}{(1-t)^T} \\ {}^{\alpha_1,f_1}(d_1)^{\eta_1,g_1}, & {}^{\alpha_2,f_2}(d_2)^{\eta_2,g_2},\ldots, & {}^{\alpha_s,f_s}(d_s)^{\eta_s,g_s}; & \end{matrix}\right]. \tag{47}$$

$$(|t|<1;\lambda\in C;T\in N).$$

And,

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\times_{r+2T}F_s\left[\begin{matrix}\Delta(T;-n),\Delta(T;\lambda+n), {}^{\delta_1,b_1}(a_1)^{\rho_1,c_1}, & {}^{\delta_2,b_2}(a_2)^{\rho_2,c_2},\ldots, {}^{\delta_r,b_r}(a_r)^{\rho_r,c_r}; \\ & Z \\ {}^{\alpha_1,f_1}(d_1)^{\eta_1,g_1}, \quad {}^{\alpha_2,f_2}(d_2)^{\eta_2,g_2},\ldots, {}^{\alpha_s,f_s}(d_s)^{\eta_s,g_s}; & \end{matrix}\right]t^n$$

$$=(1-t)^{-\lambda}\times_{r+2T}F_s\left[\begin{matrix}\Delta(2T;\lambda), {}^{\delta_1,b_1}(a_1)^{\rho_1,c_1}, & {}^{\delta_2,b_2}(a_2)^{\rho_2,c_2},\ldots, {}^{\delta_r,b_r}(a_r)^{\rho_r,c_r}; \\ & z\left(\dfrac{-4t}{(1-t)^2}\right)^n \\ {}^{\alpha_1,f_1}(d_1)^{\eta_1,g_1}, \quad {}^{\alpha_2,f_2}(d_2)^{\eta_2,g_2},\ldots, {}^{\alpha_s,f_s}(d_s)^{\eta_s,g_s}; & \end{matrix}\right]. \tag{48}$$

$$(|t|<1;\lambda\in C;T\in N),$$

and,

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\times_{r+T}F_{s+T}\left[\begin{matrix}\Delta(T;-n), {}^{\delta,b}(a_1)^{\rho,c}, {}^{\delta_2,b_2}(a_2)^{\rho_2,c_2},\ldots, {}^{\delta_r,b_r}(a_r)^{\rho_r,c_r}; & \\ & Z \\ \Delta(T;1-\lambda-n), {}^{\alpha_1,f_1}(d_1)^{\eta_1,g_1}, {}^{\alpha_2,f_2}(d_2)^{\eta_2,g_2},\ldots, {}^{\alpha_s,f_s}(d_s)^{\eta_s,g_s}; & \end{matrix}\right]t^n$$

$$=(1-t)^{-\lambda}\quad {}_rF_s\left[\begin{matrix}{}^{\delta_1,b_1}(a_1)^{\rho_1,c_1}, \quad {}^{\delta_2,b_2}(a_2)^{\rho_2,c_2},\ldots, {}^{\delta_r,b_r}(a_r)^{\rho_r,c_r}; & \\ & zt^T \\ {}^{\alpha_1,f_1}(d_1)^{\eta_1,g_1}, \quad {}^{\alpha_2,f_2}(d_2)^{\eta_2,g_2},\ldots, {}^{\alpha_s,f_s}(d_s)^{\eta_s,g_s}; & \end{matrix}\right]. \tag{49}$$

$$(|t|<1;\lambda\in C;T\in N).$$

Proof: The proof of Theorem 7 is much akin to that of Theorem 6.
Finally, we choose to state the simplest consequences of the generating functions (45) and (47) when T=1 as Corollary 6 below.

Corollary 6: Each of the following generating functions holds true:

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\times_{r+1}F_s\left[\begin{matrix}\lambda+n, {}^{\delta_1,b_1}(a_1)^{\rho_1,c_1}, & {}^{\delta_2,b_2}(a_2)^{\rho_2,c_2},\ldots, {}^{\delta_r,b_r}(a_r)^{\rho_r,c_r}; \\ & Z \\ {}^{\alpha_1,f_1}(d_1)^{\eta_1,g_1}, \quad {}^{\alpha_2,f_2}(d_2)^{\eta_2,g_2},\ldots, {}^{\alpha_s,f_s}(d_s)^{\eta_s,g_s}; & \end{matrix}\right]t^n$$

$$=(1-t)^{-\lambda}\times_{r+1}F_s\left[\begin{matrix}\lambda, {}^{\delta_1,b_1}(a_1)^{\rho_1,c_1}, {}^{\delta_2,b_2}(a_2)^{\rho_2,c_2},\ldots, {}^{\delta_r,b_r}(a_r)^{\rho_r,c_r}; & \\ & \dfrac{z}{1-t} \\ {}^{\alpha_1,f_1}(d_1)^{\eta_1,g_1}, \quad {}^{\alpha_2,f_2}(d_2)^{\eta_2,g_2},\ldots, {}^{\alpha_s,f_s}(d_s)^{\eta_s,g_s}; & \end{matrix}\right]. \tag{50}$$

$$(|t|<1;\lambda\in C).$$

and

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\times_{r+1}F_s\left[\begin{matrix}-n,{}^{\delta_1,b_1}(a_1)^{\rho_1,c_1}, & {}^{\delta_2,b_2}(a_2)^{\rho_2,c_2},\dots,{}^{\delta_r,b_r}(a_r)^{\rho_r,c_r}; \\ & Z \\ {}^{\alpha_1,f_1}(d_1)^{\eta_1,g_1}, & {}^{\alpha_2,b_2}(d_2)^{\eta_2,g_2},\dots,{}^{\alpha_s,f_s}(d_s)^{\eta_s,g_s};\end{matrix}\right]t^n$$

$$=(1-t)^{-\lambda}\times_{r+1}F_s\left[\begin{matrix}\lambda,{}^{\delta_1,b_1}(a_1)^{\rho_1,c_1}, & {}^{\delta_2,b_2}(a_2)^{\rho_2,c_2},\dots,{}^{\delta_r,b_r}(a_r)^{\rho_r,c_r}; \\ & \frac{-zt}{1-t} \\ {}^{\alpha_1,f_1}(d_1)^{\eta_1,g_1}, & {}^{\alpha_2,f_2}(d_2)^{\eta_2,g_2},\dots,{}^{\alpha_s,f_s}(d_s)^{\eta_s,g_s};\end{matrix}\right]. \tag{51}$$

$$(|t|<1;\lambda\in C).$$

and,

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\times_{r+2}F_s\left[\begin{matrix}-n,\lambda+n,{}^{\delta_1,b_1}(a_1)^{\rho_1,c_1}, & {}^{\delta_2,b_2}(a_2)^{\rho_2,c_2},\dots,{}^{\delta_r,b_r}(a_r)^{\rho_r,c_r}; \\ & Z \\ {}^{\alpha_1,f_1}(d_1)^{\eta_1,g_1}, & {}^{\alpha_2,f_2}(d_2)^{\eta_2,g_2},\dots,{}^{\alpha_s,f_s}(d_s)^{\eta_s,g_s};\end{matrix}\right]t^n$$

$$=(1-t)^{-\lambda}\times_{r+2}F_s\left[\begin{matrix}\Delta(2;\lambda),{}^{\delta_1,b_1}(a_1)^{\rho_1,c_1}, & {}^{\delta_2,f_2}(a_2)^{\rho_2,c_2},\dots,{}^{\delta_r,b_r}(a_r)^{\rho_r,c_r}; \\ & \frac{-4zt}{(1-t)^2} \\ {}^{\alpha_1,f_1}(d_1)^{\eta_1,g_1}, & {}^{\alpha_2,f_2}(d_2)^{\eta_2,g_2},\dots,{}^{\alpha_s,f_s}(d_s)^{\eta_s,g_s};\end{matrix}\right]. \tag{52}$$

$$(|t|<1;\lambda\in C).$$

and,

$$\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\times_{r+1}F_{s+1}\left[\begin{matrix}-n,{}^{\delta_1,b_1}(a_1)^{\rho_1,c_1}, & {}^{\delta_2,b_2}(a_2)^{\rho_2,c_2},\dots,{}^{\delta_r,b_r}(a_r)^{\rho_r,c_r}; \\ & Z \\ 1-\lambda-n, & {}^{\alpha_1,b_1}(d_1)^{\eta_1,g_1}, {}^{\alpha_2,b_2}(d_2)^{\eta_2,g_2},\dots,{}^{\alpha_s,f_s}(d_s)^{\eta_s,g_s};\end{matrix}\right]t^n$$

$$=(1-t)^{-\lambda}\times_{r}F_s\left[\begin{matrix}{}^{\delta_1,b_1}(a_1)^{\rho_1,c_1}, & {}^{\delta_2,b_2}(a_2)^{\rho_2,c_2},\dots,{}^{\delta_r,b_r}(a_r)^{\rho_r,c_r}; \\ & zt \\ {}^{\alpha_1,f_1}(d_1)^{\eta_1,g_1}, & {}^{\alpha_2,f_2}(d_2)^{\eta_2,g_2},\dots,{}^{\alpha_s,f_s}(d_s)^{\eta_s,g_s};\end{matrix}\right]. \tag{53}$$

$$(|t|<1;\lambda\in C).$$

Finally, since

$$\lim_{|\lambda|\to\infty}(\lambda)_n(\frac{z}{\lambda})^n=z^n=\lim_{|\mu|\to\infty}\frac{(\mu z)^n}{(\mu)_n}\quad(\lambda,\mu\in C;n\in N_0),$$

A limit case of the generating function(48) when t is replaced by $\frac{t}{\lambda}$ and $|\lambda|\to\infty$ yields the following exponential generating function:

$$\sum_{n=0}^{\infty}{}_{r+1}F_s\left[\begin{matrix}-n,{}^{\delta_1,b_1}(a_1)^{\rho_1,c_1}, & {}^{\delta_2,b_2}(a_2)^{\rho_2,c_2},\dots,{}^{\delta_r,b_r}(a_r)^{\rho_r,c_r}; \\ & Z \\ {}^{\alpha_1,f_1}(d_1)^{\eta_1,g_1}, & {}^{\alpha_2,f_2}(d_2)^{\eta_2,g_2},\dots, {}^{\alpha_s,f_s}(d_s)^{\eta_s,g_s};\end{matrix}\right]\frac{t^n}{n!}$$

$$=(e)^t{}_rF_s\left[\begin{matrix}{}^{\delta_1,b_1}(a_1)^{\rho_1,c_1}, & {}^{\delta_2,b_2}(a_2)^{\rho_2,c_2},\dots,{}^{\delta_r,b_r}(a_r)^{\rho_r,c_r}; \\ & -zt \\ {}^{\alpha_1,f_1}(d_1)^{\eta_1,g_1},{}^{\alpha_2,f_2}(d_2)^{\eta_2,g_2},\dots,{}^{\alpha_s,f_s}(d_s)^{\eta_s,g_s}; & \end{matrix}\right]. \tag{54}$$

$$(|t|<1;\lambda\in C).$$

5. REFERENCES

[1] M. Abramowitz,I.A.Stegun (eds), Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, Applied Mathematics Series 55,National Bureau of Standards, Washington, D.C., 1964; Reprinted by Dover Publications, New York, 1965.

[2] G.E. Andrews, R. Askey, R. Roy, Special Functions, Encyclopedia of Mathematics and Its Applications, Vol. 71, Cambridge University Press, Cambridge, London and New York, 1999.

[3] A. Hasanov, H.M. Srivastava, M. Turaev, Decomposition formulas for some triple hypergeometric functions, J. Math. Anal. Appl. 324(2006) 955-969.

[4] W. Magnus, F. Oberhettinger, R.P. Soni, Formulas and Theorems for the Special Functions of Mathematical Physics, Third Enlarged Edition, Die Grundlehren der Mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtingung der Anwendungsgebiete, vol. 52, Springer-Verlag, Berlin, Heidelberg and New York, 1966.

[5] F.W.J. Olver, D.W. Lozier, R.F. Boisvert and C.W. Clark (eds), NIST Handbook of Mathematical Functions [With 1 CD-ROM], U.S. Department of Commerce, National Institute of Standards and Technology, Washington, D.C., 2010; Cambridge University Press, Cambridge, London and New York, 2010 (see also AS).

[6] M.A. Özarslan, Some remarks on extended hypergeometric, extended concuent hypergeometric and extended Appell's functions, J.Comput. Anal. Appl. 14 (2012) 1148-1153.

[7] M.A. Özarslan, E. Özergin, Some generating relations for extended hypergeometric functions via generalized fractional derivative operator, Math. Comput. Modelling 52 (2010) 1825-1833.

[8] E. Özergin, M.A. Özarslan, A. Altn, Extension of gamma, beta and hypergeometric functions, J. Comput. Appl. Math. 235 (2011)4601-4610.

[9] A.P. Prudnikov, Yu. A. Brychkov, O.I. Marichev, Integrals and Series, Vol. 1: Elementary Functions, Gordon and Breach, New York, 1986; Reprintedby Taylor and Francis, London, 1998.

[10] E.D. Rainville, Special Functions, Macmillan Company, New York, 1960; Reprinted by Chelsea publishing Company, Bronx, New York,1971.

[11] H.M. Srivastava, Aysegul Cetinkaya,I. Onur Kiymaz, A certain generalized Pochhammer symbol and its applications to hypergeometric functions, Applied Mathematics and Computation 226(2014) 484-491.

[12] H.M. Srivastava, H.L. Manocha, A Treatise on Generating Functions, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1984.

[13] H.M. Srivastava, R.K. Saxena, C. Ram, A unifed presentation of the gamma-type functions occurring in diffration theory and associated probability distributions, Appl. Math. Comput. 162 (2005) 931-947.

[14] Kuldeep Singh Gehlot, Properties of Ultra Gamma Function, arXiv: 1704.08189v1[math.CA] 15 April 2017.

[15] Kuldeep Singh Gehlot ,Differential Equation of K-Bessel's Function and its Properties,Nonl. Analysis and Differential Equations, Vol. 2, 2014,61-67.

[16] Kuldeep Singh Gehlot, Two Parameter Gamma Function and it's Properties, arXiv:1701.01052v1[math.CA] 3 Jan 2017.

[17] K.S. Gehlot, SD Purohit, Fractional Calculus of K-Bessel's function, Acta Universitatis Apulensis 38,273-278,2014.

[18] K.S. Gehlot, SD Purohit, Integral representations of the K-Bessel's function, Honam Math.J 38, 17-23,2016.

[19] G.N. Watson, A Treatise on the Theory of Bessel Functions, Second edition, Cambridge University Press, Cambridge, London and New York, 1944.

