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Properties of GR-regular spaces in topology

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ABSTRACT

In 2011, S. Bhattacharya (Halder) has introduced the concepts of gr-closed sets in topological spaces. In 2012, S. I. Mahmood has defined and studied the concepts of gr-continuity, gr-irresoluteness, perfect gr-continuity and Tgr– spaces. The purpose of this paper is to investigate and study the concepts of gr-regular spaces, Gr-regular spaces, GR-regular spaces along with some allied gr-continuity and some allied gr-regularity axioms in topology.

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Keywords— *Regular open sets, Regular closed sets, G-closed sets and RG-closed sets, GR-closed sets, GR-continuity, GR-irresoluteness*

1. INTRODUCTION

Levine [5] generalized the closed set to generalized closed set (g-closed set) in topology for the first time. Since then it is noticed that some of the weaker forms of closed sets have been generalized. In 1993, Palaniappan and Rao [9] have introduced and studied the notions of rg-closed sets, rg-open sets, rg-continuous functions, rg-irresoluteness.In 2011, S.Bhattacharya (Halder) [4] introduced the concept of generalized regular closed sets in topology and then in 2012, S.I.Mahmood [6] has defined and studied the concepts of gr-continuity, gr-irresoluteness, perfect gr-continuity and T_{gr} – spaces.

2. PRELIMINARIES

Throughout this paper (X,τ), (Y,σ), (Z,γ) (or simply X, Y, and Z) always means topological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of space X. We denote the closure of A and the interior of A by Cl(A) and Int(A) respectively.

Definition 2.1: The subset of A of X is said to be:
(i) Regular open (in brief,r-open) if A = IntCl(A).
(ii) Regular closed (in brief,r-closed) if A = ClInt(A).

Definition 2.2: The subset of A of X is said to be.
(i) semi-pre open[2]set , if $A \subset Cl(Int(Cl(A)))$
(ii) b-open [3] if $A\subset$ ClInt(A) $\cup$ IntCl(A).
(iii) δ-closed [10] if A =δ Cl(A) , whereδ Cl(A) = $\{x\in X : IntCl(U)\cap A\neq\emptyset$, U is open set and $x\in U\}$
The complement of semipre-open (resp.b-open ,δ-closed) set is called semipre-closed [2] (resp.b-closed [3] , δ-open [10]) set of a space X.

Definition 2.3 [2]: The intersection of all semipre-closed sets of X containing subset A is called the semipre-closure of A and is denoted by spCl (A).

Definition 2.4 [10]: The intersection of all regular closed(resp. δ-closed) sets containing set A is called the regular closure(resp.δ-closure) of A and is denoted by rCl(A) (resp.δCl(A)) .

Definition 2.5 [3]: The intersection of all b-closed sets containing set A is called the b- closure of A and is denoted by bCl(A). Similarly, spInt(A) , rInt(A) ,bInt(A),δ Int(A) can be defined.

Definition 2.6: A subset A of a space X is called:

(i) Generalized closed set (in brief, g-closed) set [5] if $Cl(A) \subset U$ whenever $A \subset U$ and U is open.
(ii) Regular generalized closed (in brief, rg-closed) set [10] if $Cl(A) \subset U$ whenever $A \subset U$ and U is regular open.
(iii) Generalized regular closed (in brief, gr-closed) set [4] if $rCl(A) \subset U$ whenever $A \subset U$ and U is open.
(iv) Generalized b- closed (in brief, gb-closed) set [1] if $bCl(A) \subset U$ whenever $A \subset U$ and U is open.
(v) Regular b-closed (in brief, rb-closed) set [8] if $rCl(A) \subset U$ whenever $A \subset U$ and U is b-open in X.

The complement of a g-closed resp. rg-closed , gr-closed ,gb-closed ,rb-closed ,δg-closed) set in X is called g-open set(resp. rg-open , gr-open , gb-open,rb-open ,δg-open) set in X.

Lemma 2.7 [4]: A subset A of a space X is gr-open in X if and only if $F \subset rInt(A)$ whenever $F \subset A$ and F is r-closed set in X.

Definition 2.8 [6]: A subset A of a space X is called gr-clopen if it is both gr-open and gr-closed.

Definition 2.9 [6]: The intersection of all gr-closed sets of X containing a set A is called the gr-closure of A and is denoted by grCl(A).
Note: if A is a gr-closed set then grCl(A) = A.

Definition 2.10 [1]: The intersection of all gb-closed sets of X containing a set A is called the gr-closure of A and is denoted by gbCl(A).
Note: if A is a gb-closed set then gbCl(A) = A .

Definition 2.11: A function $f: X \to Y$ is said to be:
(i) gr-continuous [6]if the inverse image of each closed set of Y is gr-closed in X.
(ii) gr-irresolute [6]if the inverse image of each gr-closed set of Y is gr-closed in X.
(iii) gb-irresolute [1]if the inverse image of each gb-closed set of Y is gb-closed in X.

Definition 2.12[7]: A space X is said to be g-regular if for each g-closed set F and each $x \in X-F$, there exist disjoint open sets U and V in X such that $x \in U$ and $F \subset V$.

3. PROPERTIES OF GR-REGULAR SPACES

The following implications are observed from [2], [5], [8] and [12]:

Lemma 3.1:
(i) Every gr-closed set is g-closed.
(ii) Every gr-closed set is rg-closed.
(iii) Every gr-closed set is gb-closed.
(iv) Every r-closed set is gr-closed.
(v) Every rb-closed set is δg-closed.
(vi) Every r-closed set is δ-closed.

We define the following:
Definition 3.2: A space X is called gr-regular if for each gr-closed set F of X and each point $x \notin F$, there exist disjoint open sets U and V of X such that $x \in U$ and $F \subset V$.
Clearly, every g-regular space is gr-regular space since by Lemma-3.1.

Definition 3.3: Let $x \in X$. A set U is called a gr-neighbourhood of point x in X if there exists a gr-open set G such that $x \in G \subset U$.

Definition 3.4: If $A \subset X$ then, any other subset G of X is called a gr-neighbourhood of the set A if G is a gr-neighbourhood of every point of A.

We, prove the following:
Theorem 3.5: A topological space X is gr-regular if and only if for each gr-closed set F of X and each point $x \in X - F$, there exist open sets U and V of X such that $x \in U$:$F \subset V$ and $Cl(U) \cap Cl(V) = \varnothing$.
Proof: Obvious.

Theorem 3.6: Let X be a topological space then the following statements are equivalent:
(i) X is gr-regular space
(ii) For each point $x \in X$ and for each gr-open neighbourhood W of x, there exists a open V set of X, such that $Cl(V) \subseteq W$
(iii) For each point of $x \in X$ and for each gr-closed set F not containing x, then there exists an open set V of X such that $Cl(V) \cap F = \varnothing$.
Proof: (i)⇒ (ii): Let W be a gr-open neighbourhood of x. Then there exists a gr-open set G such that $x \in X \subseteq W$. Since (X-G) is gr-closed set and $x \notin (X-G)$, by hypothesis there exist open sets U and V such that $(X-G) \subseteq U$, $x \in V$ and $U \cap V = \varnothing$ and so $V \subset (X-U)$ Now $Cl(V) \subseteq Cl(X-U)$ and $(X-G) \subseteq U$ implies $(X-U) \subseteq G \subseteq W$. Therefore $Cl(V) \subseteq W$.

(ii)⇒**(i)**: Let F be any gr-closed set of $x \notin F$. Then $x \in X-F$ and *(X-F)* is gr-open and so *(X−F)* is a gr-open neighbourhood of x. By hypothesis, there exists an open V of x such that $x \in V$ and $Cl(V) \subseteq (X-F)$ which implies $F \subseteq (X-Cl(V)))$ Then $X-Cl(V)))$ is an open set containing F and $V \cap (X-Cl(V))=\varnothing$. Therefore X is gr-regular space.
(ii)⇒(iii): Let $x \notin X$ and F be a gr-closed set such that $x \in F$. Then (X-F) is a gr-open neighbourhood of x and by hypothesis there exists an open set V of x such that $Cl(V) \subseteq (X-F)$ and therefore $Cl(V) \cap F = \varnothing$.
(iii)⇒**(ii)**: Let $x \in X$ and W be a gr-open neighbourhood of x then there exists a gr-open set G such that $x \in G \subseteq W$.Since *(X-G)* is gr-closed and $x \notin (X-G)$ by hypothesis there exists an open set V of x such that $Cl(V) \cap (X-G)=\varnothing$. Therefore $Cl(V) \subseteq G \subseteq W$.

We define the following:
Definition 3.7: A space X is called Gr-regular if for each closed set F of X and each point $x \notin F$, there exist disjoint gr-open sets U and V of X such that $x \in U$ and $F \subset V$.

Definition 3.8: A space X is called G-regular if for each closed set F of X and each point $x \notin F$, there exist disjoint g-open sets U and V of X such that $x \in U$ and $F \subset V$.
Clearly, every Gr-regular space is G-regular space since by Lemma-3.1.

Now, we state the following:
Theorem 3.9: Let X be a topological space then the following statements are equivalent:
(i) X is Gr-regular space
(ii) For each point $x \in X$ and for each open neighbourhood W of x, there exists a gr-open V set of X, such that $grCl(V) \subseteq W$
(iii) For each point of $x \in X$ and for each closed set F not containing x, then there exists a gr-open set V of X such that $grCl(V) \cap F = \varnothing$.

Definition 3.10: A space X is called δ gr-regular if for each δ-closed set F of X and each point $x \notin F$, there exist disjoint gr-open sets U and V of X such that $x \in U$ and $F \subset V$.
Clarly, every Gr-regular space is δgr-regular space since every δthe -closed set is closed set.

Lemma 3.11: For a space X the following is true. A space X is δgr-regular if for each δ-open set U of X containing x,there exists gr-open set V such that $x \in V \subset grCl(V) \subset U$.
The proof is obvious and hence omitted.

Definition 3.12: A space X is called (gb, gr)-regular if for each gb-closed set F of X and each point $x \notin F$, there exist disjoint gr-open sets U and V of X such that $x \in U$ and $F \subset V$.

Definition 3.13: A space X is called GR-regular if for each gr-closed set F of X and each point $x \notin F$, there exist disjoint gr-open sets U and V of X such that $x \in U$ and $F \subset V$.
Clearly, every (gb, gr)-regular space is GR-regular space since every gr-closed set is gb-closed set. Also, every δgr-regular space is GR-regular as every δthe -closed set is gr-closed set.

Theorem 3.14: For a space X the following are equivalent:
(i) X is GR-regular space.
(ii) For each $x \in X$ and for each gr-open set U containing x there exists a gr-open set V containing x such that $x \in V \subset grCl(V) \subset U$.
(iii) For each gr-closed set F of X, $\cap \{grCl(V) : F \subset V$ and V is gr-open in X$\} = F$.
(iv) For each non-empty subset, A of X and each gr-open set U of X if $A \cap U \neq \varnothing$ and $grCl(V) \subset U$.
(v) For each non-empty subset A of X and each gr-closed set F of X if $A \cap F = \varnothing$, then there exist gr-open sets V and W of X such that $A \cap V \neq \varnothing$, $F \subset W$ and $V \cap W = \varnothing$.

Proof: (i)⇒(ii) : Let X be GR-regular space. Let $x \in X$ and U be gr-open set containing X ⇒ X-U is gr-closed set such that $x \notin X-U$. Therefore by (i) ,the re exist two gr-open sets $= \varnothing \Rightarrow grCl(V) \subset X-W \subset U$. Thus, $x \in V \subset grCl(V) \subset U$.
(ii)⇒(iii): Let F be a gr-closed subset of X and $x \notin F$ then X-F is gr-open set containing x. Then by (ii), there exists gr-open set U such that $x \in U \subset grCl(U) \subset X-F \Rightarrow F \subset X-grCl(U) \subset X-U$ i.e., $F \subset V \subset X-U$, where $V=X-grCl(U)$ which is gr-open set and $x \notin V$ that implies $x \notin grCl(V) \Rightarrow x \notin \cap \{grCl(V): F \subset V$ and V is gr-open set in X$\}$. Hence, $\cap \{grCl(V): F \subset V$ and V is gr-open set in X$\} = F$.
(iii)⇒(iv) : Let A be a subset of X and U be a gr-open set such that $A \cap U \neq \varnothing$. This implies that there exists a point $x_0 \in X$ such that $x_0 \in A \cap U$. Therefore, X-U is gr-closed set not containing $x_0 \Rightarrow x_0 \in grCl(X-U)$. Then by (iii), there exists a gr-open set W in X such that $X-U \subset W \Rightarrow x_0 \notin grCl(W)$. Now, put $V = X-grCl(W)$, then V is gr-open set containing $x_0 \Rightarrow A \cap V \neq \varnothing$ and $grCl(V) \subset grCl(X-grCl(W)) \subset grCl(X-W)$. Therefore, $grCl(V) \subset grCl(X-U) \subset U$.
(iv)⇒ (v) : Let A be a non-empty subset of X and F be gr-closed set such that $A \cap F = \varnothing$.Then, X-F is gr-open set in X and $A \cap (X-F) \neq \varnothing$. Then by (iv), there exists gr-open set V such that $A \cap V \neq \varnothing$ and $grCl(V) \subset X-F$. Now, put $W = X-grCl(V)$ then W is gr-open set in X such that $F \subset$ and $W \cap V = \varnothing$.
(v)⇒(i): Let $x \in X$ be an arbitrary point and F be gr-closed set not containing x. Let A be a non-empty gr-open set containing x then by (v), there exist disjoint gr-open sets V and W such that $F \subset W$ and $A \cap V \neq \varnothing \Rightarrow x \in V$. Thus , X is GR-regular space.

We, define the following:
Definition 3.15: A subset A of a space X is called gr-clopen if it is both gr-open set and gr-closed set in X.

Next, we prove the folloing:
Lemma 3.16: A space X is GR-regular if and only for each gr-open set U of X and each $x \in U$, there exists a gr-clopen set V such that $x \in V \subset U$.
The proof follows from Th.3.22 and Def.3.23.

We, define the following:
Definition 3.17: A bijective function f: $X \to Y$ is called gr-homeomorphism if both f and f^{-1} are gr-irresolute functions.

Next, we prove the following.
Lemma 3.18: Any gr-homeomorphic image of GR-regular space is GR-regular.
Proof: Obvious.

We, define the following:
Definition 3.19: A space X is said to be gr-T_1 – space if for each pair of distinct points $x,y \in X$, there exist two gr-open sets U and V such that $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$.

Definition 3.20: A space X is said to be gr-T_2 – space if for each pair of distinct points $x,y \in X$, there exist two disjoint gr-open sets U and V such that $x \in U$ and $y \in V$.
Clearly, every gr-T_2 –space is gr-T_1 –space.

We, state the following:
Lemma 3.21: A space X is gr-T_1-space iff for each $x \in X$, $\{x\}$ is gr-closed set in X.

Lemma 3.22: If X be gr-T_1 and GR-regular space then it is gr-T_2-space.

Next, we define the following:
Definition 3.23: A space X is called gb-regular if for each gb-closed set F of X and each point $x \notin F$, there exist disjoint open sets U and V of X such that $x \in U$ and $F \subset V$.

Definition 3.24: A space X is called Gb-regular if for each closed set F of X and each point $x \notin F$, there exist disjoint gb-open sets U and V of X such that $x \in U$ and $F \subset V$.

Definition 3.25: A space X is called GB-regular if for each gb-closed set F of X and each point $x \notin F$, there exist disjoint gb-open sets U and V of X such that $x \in U$ and $F \subset V$.
Clearly, every GB-regular space is Gb-regular.

We, prove the following:
Lemma 3.26: Let X be a GB-regular space if for each $x \in X$ and for each gb-open set U containing x ,there exists a gr-open set V containing x such that $x \in V \subset gbCl(V) \subset U$.
The easy proof of the lemma is omitted.

Next, we define the following:
Definition 3.27: A subset A of a space X is called gb-clopen if it is both gb-open sets and gb-closed set in X.

Now, we prove the following:
Theorem 3.28: A space X is GB-regular iff for each gb-open set U of X and each $x \in U$, There exists a gb-clopen set V such that $x \in V \subset U$.
Proof: Similar to Th.3.24.

We, define the following:
Definition 3.29: A bijective function f: $X \to Y$ is called gb-homeomorphism if bothf and f^{-1} are gb-irresolute functions.

Next, we prove the following:
Lemma 3.30: Any gb-homeomorphic image of GB-regular space is GB-regular.
Proof: Obvious.

We, define the following:
Definition 3.31: A space X is said to be gb-T_1 – space if for each pair of distinct points $x,y \in X$, there exist two gb-open sets U and V such that $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$.

Definition 3.32: A space X is said to be gb-T_2 – space if for each pair of distinct points $x,y \in X$, there exist two disjoint gb-open sets U and V such that $x \in U$ and $y \in V$.
Clearly, every gb-T_2 –space is gb-T_1 –space.

We, state the following:

Lemma 3.29: A space X is gb-T_1-space iff for each $x \in X$, $\{x\}$ is gb-closed set in X.

Lemma 3.30: If X be gb-T_1 and GB-regular space then it is gb-T_2-space.

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