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Introduction to sets of linear regression models – A brief review

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ABSTRACT

Seemingly Unrelated Regression Equations (SURE) model is the generalization of the Linear regression model. Zellner (1962) proposed the SURE model and its various associated estimators, test statistics and generalizations have generated a substantial body of literature on sets of linear regression models. Here we specify the SURE models with the assumptions and also explains different estimation methods such as Ordinary Least Squares(OLS), Generalized Least Squares (GLS), Zellner's Feasible Generalized Least Squares (FGLS), Seemingly Unrelated Unrestricted Residual (SUUR) and Seemingly Unrelated Restricted Residual (SURR) have been explained with their properties.

Keywords: *Linear Regression Model, Seemingly Unrelated Regression Equation Model, Ordinary Least Squares, Generalized Least Squares, Feasible Generalized Least Squares, Seemingly Unrelated Unrestricted Residual, Seemingly Unrelated Restricted Residual*

1. INTRODUCTION

Sets of regression equations occur in the various branches of economic theory. The sets of linear regression models have wide applications in the theory of economics. For instance, the various problems under the theory of consumer behavior and the theory of production functions can be analyzed through the sets of linear regression models.

Consider a set of individual multiple linear regression equations, each explaining some economic phenomenon. This set of regression equations is said to comprise a simultaneous equations model if one or more of the regressor's in one or more of the equations is itself the dependent variable associated with another equation in the full system.

On the other hand, suppose that none of the variables in the system are simultaneously both explanatory and dependent in nature. There may still be interactions between the individual equations if the random disturbances associated with at least some of the different equations are correlated with each other. This means that the equations may be linked statistically, even though not structurally through the joint ness of the disturbances distribution and through the non-diagonally of the associated variance covariance matrix.

Zellner (1962) discussed this possibility a referred it as "Seemingly Unrelated Regression Equations (SURE)" to reflect the fact that the individual equations are in fact related to one another, even though superficially they may not seem to be. Zellner (1962), in his path-breaking article, motivated this point and proposed the SUR approach for estimating a system of linear regression equations in which the disturbances are contemporaneously correlated across the equations but not auto correlated. He derived the asymptotically efficient estimators of the parameters of the SURE model. Zellner's article is the basis for the development of statistical inference in sets of linear regression models.

2. SPECIFICATION OF THE SEEMINGLY UNRELATED REGRESSION EQUATIONS (SURE) MODEL

Consider a system of M – multiple linear regression equation is

$$Y_{ij} = \sum_{p=1}^{k_i} X_{ijp}\beta_{ip} + \in_{ij} \ , \quad i = 1,2,\cdots\cdots,m \text{ and } j = 1,2,\cdots\cdots,n \tag{2.1}$$

Where Y_{ij} is the j^{th} observation on the i^{th} dependent variable.

X_{ijp} is the j^{th} observation on the P^{th} explanatory variable appearing in the i^{th} regression equation

β_{ip} is the regression coefficient associated with X_{ijp} at each observation and $\in_{ij}$ is the j^{th} value of the random disturbance term associated with the i^{th} equation of the model.

In matrix notation, this M – equation model may be expressed more compactly as

$$Y_i = X_i\beta_i + \in_i \ , \ i = 1,2,\cdots\cdots,m \tag{2.2}$$

The entire system of equation (2.2) may be conveniently written as:

$$\begin{bmatrix} Y_1 \\ Y_2 \\ . \\ . \\ Y_m \end{bmatrix} = \begin{bmatrix} X_1 & 0 & . & . & . & 0 \\ 0 & X_2 & . & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & X_m \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ . \\ . \\ \beta_m \end{bmatrix} + \begin{bmatrix} \in_1 \\ \in_2 \\ . \\ . \\ \in_m \end{bmatrix}$$

By treating each of the n equations as a classical linear regression model, the crucial assumptions about the system may be written as

$(i) E[\in_i] = 0, \forall i = 1,2,\cdots\cdots,m$

$(ii) E[\in_i \in_i^1] = \sigma_{ii} I_n, \forall i = 1,2,\cdots\cdots,m$

Where σ_{ii} represent the variance of the random disturbance in the i^{th} equation for each observation in the sample and I_n is the identity matrix of order n.

$(iii) E[\in_i \in_j^1] = \sigma_{ij} I_n, \forall i \neq j = 1,2,\cdots\cdots,m$

Where σ_{ij} represents the covariance between the disturbances of the i^{th} and j^{th} equations for each observation in the sample. Thus, the disturbances in different linear regression equations are correlated.

$(iv) \lim\limits_{n\to\alpha} \left[\dfrac{X_i^1 X}{n}\right] = Q_{ii}$ and, $i, j = 1,2,\cdots\cdots,n$

Where Q_{ii} and Q_{ij} are non-singular with fixed and finite elements.

$(v) \in_i$ follows a multivariate probability distribution.

3. THE ORDINARY LEAST SQUARES (OLS) ESTIMATION OF SURE MODEL

Consider the system of SUR equations in matrix notation as

$$Y_{(nmx1)} = X_{(nmxk^*)}\beta_{(k^*x1)} + \in_{(nmx1)} \tag{3.1}$$

$$\text{With } E(\in) = 0 \text{ and } E[\in\in^1] = [\Sigma \otimes I_n] = \psi \tag{3.2}$$

The OLS method applied to the combined system (3.1) is identical to the OLS method applied to each equation separately. For i^{th} the linear regression model, the OLS estimator β_i is given by

$$\hat{\beta} = (X_i^1 X_i) X_i^1 Y_i \ , \ i = 1,2,\cdots\cdots,n \tag{3.3}$$

and hence, the OLS estimator β is given by $\hat{\beta} = (X^1 X) X^1 Y$ (3.4)

$$\Rightarrow \begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \\ . \\ . \\ \hat{\beta}_n \end{bmatrix} = \begin{bmatrix} (X_1^1 X_1)^{-1} X_1^1 Y_1 \\ (X_2^1 X_2)^{-1} X_1^1 Y_2 \\ . \\ . \\ (X_m^1 X_m)^{-1} X_m^1 Y_m \end{bmatrix} \tag{3.5}$$

It should be noted that the OLS estimator $\hat{\beta}$ is not an optimal estimator for β because of the following two reasons :

(i) $\in_1, \in_2, \cdots\cdots, \in_n$ may not be homoscedastic

(ii) $\in_1, \in_2, \cdots\cdots, \in_n$ may be correlated.

It can be seen that $\hat{\beta}$ is an unbiased estimator for β and its variance co-variance matrix is given by

$$V(\hat{\beta}) = (X^1 X)^{-1} X [\Sigma \otimes I_n] X (X^1 X)^{-1} \tag{3.6}$$

Thus, the OLS estimator $\hat{\beta}$ in (3.4) ignores the joint prior variance co-variance matrix ψ and the non-zero correlations between the disturbances.

4. THE GENERALIZED LEAST SQUARES (GLS) ESTIMATION OF SURE MODEL

To take account of the variance co-variance matrix of the disturbances, one may obtain the GLS estimator or Aitken estimator of β for a SURE model as:

$$\tilde{\beta} = [X^1 \psi^{-1} X] X^1 \psi^{-1} Y \tag{4.1}$$

or $\tilde{\beta} = \left[X^1\left(\Sigma^{-1} \otimes I_n\right)X\right]^{-1} X^1\left(\Sigma^{-1} \otimes I_n\right)Y$ (4.2)

It can be seen that the GLS estimator $\tilde{\beta}$ is an unbiased estimator for β and its variance co-variance matrix is given by

$$Var\left(\tilde{\beta}\right) = \left[X^1\psi^{-1}X\right]^{-1} \text{ or } Var\left(\tilde{\beta}\right) = \left[X^1\left(\Sigma^{-1} \otimes I_n\right)X\right]^{-1} \quad (4.3)$$

Consider $$Var\left(\hat{\beta}\right) - Var\left(\tilde{\beta}\right) = \left(X^1X\right)^{-1}\left(X^1\psi^{-1}X\right)\left(X^1X\right)^{-1} - \left(X^1\psi^{-1}X\right)^{-1} \quad (4.4)$$

By defining $W = \left(X^1X\right)^{-1}X^1 - \left(X^1\psi^{-1}X\right)^{-1}X^1\psi^{-1}$

Such that, one may write (4.4) as

$$Var\left(\hat{\beta}\right) - Var\left(\tilde{\beta}\right) = W\psi W^1 \quad (4.5)$$

Since, ψ is positive definite. $W\psi W^1$ is at least positive definite, and hence, the GLS estimator $\tilde{\beta}$ is at least as is OLS estimator $\hat{\beta}$, when estimating β in the SURE model (4.1).

ZELLNER'S FEASIBLE GLS ESTIMATION OF SURE MODEL:

Consider the SURE model (4.1) as:

$$Y = X\beta + \in \quad (4.6)$$

with $$E[\in] = 0 \text{ and } E\left[\in\in^1\right] = \left(\Sigma \otimes I_n\right) = \psi \quad (4.7)$$

The GLS estimator β is given by

$$\tilde{\beta} = \left(X^1\psi^{-1}X\right)^{-1}X^1\psi^{-1}Y \quad (4.8)$$

or $$\tilde{\beta} = \left[X^1\left(\Sigma^{-1} \otimes I_n\right)X\right]^{-1} X^1\left(\Sigma^{-1} \otimes I_n\right)Y \quad (4.9)$$

In general Σ and hence ψ will be unobservable. Thus, $\tilde{\beta}$ may not be an operational or feasible estimator of β, Zellner (1962) proposed an estimator of β, based on, but with Σ replaced by an observable (mxm) matrix, S, which is a consistent estimator for Σ based on restricted residuals obtained by estimating each equation by the method of least squares. In other words, the elements of S are the estimators of the corresponding elements of.

Now, the Zellner's feasible GLS estimator β is given by

$$\tilde{\beta}^* = \left[X^1\left(S^{-1} \otimes I_n\right)X\right]^{-1} X^1\left(S^{-1} \otimes I_n\right)Y$$

Here $S = \left(\left(S_{ij}\right)\right)$ is non-singular, where S_{ij} is an estimator of σ_{ij}

There are many possible choices of S. Among them, two popular ways of obtaining the S_{ij} 's are based on residuals obtained by the preliminary application of OLS estimation in one way or another. These two choices of S lead to two estimators namely

(i) Seemingly Unrelated Unrestricted Residuals (SUUR) estimator

(ii) Seemingly Unrelated Restricted Residuals (SURR) estimator.

5. THE SEEMINGLY UNRELATED UNRESTRICTED RESIDUALS (SUUR) ESTIMATOR

Suppose Y and $\in$ be (nxm) matrices with typical column vectors Y_i and $\in_i, (i = 1,2,\cdots\cdots,m)$ respectively. Let Z be the (nxk) matrix of n observations on each of the K distinct explanatory variables in the model. Also, let B be a (kxm) matrix of unknown regression coefficients, then the multivariate regression model is given by

$$Y = ZB + \in \quad (5.1)$$

$$E[\in] = 0 \text{ and } E\left[\frac{\in^1\in}{n}\right] = \Sigma \quad (5.2)$$

If $\in_i$ is the i^{th} column vector of $\in$ and β_i be the i^{th} column vector of the coefficient matrix, then the m equations may be written as

$$Y_i = Z\beta_i + \in_i \quad , \quad i = 1,2,\cdots\cdots,m \quad (5.3)$$

It should be noted that in the SURE model, all are are sub-matrices of Z and

$$X_i = ZJ_i \;, i = 1,2,\cdots\cdots,m \quad (5.4)$$

Where J_i is a (kxk_i)matrix with elements taking the value zero or unity, as appropriate?

By estimating each equation of (5.4) using OLS, one can obtain the $(nx1)$unrestricted residual vectors as

$$e_i = \left[Y_i - Z\left(Z^1Z\right)^{-1}Z^1Y_i\right] = M_zY_i \;, i = 1,2,\cdots\cdots,m$$

Where $M_z = \left[I_n = Z\left(Z^1Z\right)^{-1}Z^1\right]$ is an idempotent matrix?

Using these residuals, the consistent estimators of the σ_{ij} 's are given by

$$S_{ij} = \frac{e_i^1 e_i}{n} \quad \text{or} \quad S_{ij} = \frac{Y_i^1 M_z Y_j}{n}, i, j = 1,2,\cdots,m \tag{5.5}$$

Consider $M_z X_i = X_i - Z\left(Z^1 Z\right)^{-1} Z^1 X_i = 0$

$$\therefore Y_i^1 M_z Y_j = \left[\beta_i^1 X_i^1 + \in_i^1\right] M_z \left[X_j \beta_j + \in_j\right] = \in_i^1 M_z \in_j \tag{5.6}$$

Thus, $E\left[S_{ij}\right] = \frac{1}{n} E\left[\in_i^1 M_z \in_j\right] = \frac{1}{n}\sigma_{ij} trace\left(M_z\right) = \left[\frac{n-k}{n}\right]\sigma_{ij} \qquad \left(\because trace\left(M_z\right) = n-k\right)$

i.e., an unbiased estimator σ_{ij} is given by $\hat{\sigma}_{ij} = \frac{e_i^1 e_j}{n-k}$.

Similarly , the minimum mean squared error (MMSE) estimator σ_{ij} can be obtained as

$$\sigma_{ij} = \frac{e_i^1 e_j}{n-k+2} \tag{5.7}$$

Hence the SUUR estimator β is given by

$$\tilde{\beta}_{UR}^* = \left[X^1\left(\hat{S}^{-1} \otimes I_n\right)X\right]^{-1} X^1\left(\hat{S}^{-1} \otimes I_n\right)Y \quad \text{, Where } \hat{S} = \left(\left(\hat{\sigma}_{ij}\right)\right)$$

6. THE SEEMINGLY UNRELATED RESTRICTED RESIDUALS (SURR) ESTIMATOR

Consider the SURE model as
$$Y = X\beta + \in \tag{6.1}$$

With
$$E\left[\in\right] = 0 \quad \text{and} \quad E\left[\in \in^1\right] = \left(\Sigma \otimes I_n\right) = \psi \tag{6.2}$$

By estimating each equation of (6.1) separately, using OLS estimation, the restricted residual vectors are given by

$$\hat{e}_i = Y_i - X_i\left(X_i^1 X_i\right)^{-1} X_i^1 Y_i = M_{X_i} Y_i, i = 1,2,\cdots\cdots,m$$

Where
$$M_{X_i} = \left[I - X_i\left(X_i^1 X_i\right)^{-1} X_i\right]$$

An alternative consistent estimator for σ_{ij} is given by

$$\tilde{S}_{ij} = \frac{\tilde{e}_i^1 \tilde{e}_j^1}{n} = \frac{Y_i^1 M_{X_i} M_{X_j} Y_j}{n} \tag{6.3}$$

It can be shown that,
$$tr\left(M_{X_i} M_{X_j}\right) = n - k_i - k_j + tr\left(\left(X_i^1 X_i\right)^{-1} X_i^1 X_j\right)\left(\left(X_j^1 X_j\right)^{-1} X_j^1 X_i\right) \tag{6.4}$$

Zellner and Haung (1962) proposed an unbiased estimator of σ_{ij} as

$$\hat{\sigma}_{ij}(R) = \tilde{S}_{ij}(R) = \frac{Y_i^1 M_{X_i} M_{X_j} Y_j}{tr\left(M_{X_i} M_{X_j}\right)} \tag{6.5}$$

Thus SURR estimator for β is given by $\hat{\beta}_{RR}^* = \left[X^1\left(\tilde{S}^{-1} \otimes I_n\right)X\right]^{-1} X^1\left(\tilde{S}^{-1} \otimes I_n\right)Y$

7. CONCLUSION

Zellner's Seemingly Unrelated Regression Equations (SURE) Model is the generalization of Linear Regression model. Different estimation methods like Ordinary Least Squares, Generalized Least Squares, Feasible Generalized Least Squares, Seemingly Unrelated Unrestricted Residuals and Seemingly Unrelated Restricted Residuals are discussed here.

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