INTERNATIONAL JOURNAL OF ADVANCE RESEARCH, IDEAS AND INNOVATIONS IN TECHNOLOGY

ISSN: 2454-132X
Impact factor: 4.295
(Volume3, Issue3)
Available online at www.ijariit.com

Matching Domination of Kronecker Product of Two Graphs

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***Abstract:** A dominating set D is called a connected dominating set, if it induces a connected sub-graph in G. Since a dominating set must contain at least one vertex from every component of G, it follows that a connected dominating set for a graph G exists if and only if G is connected. The minimum of cardinalities of the connected dominating sets of G is called the connected domination number of G and is denoted by $_{c}(G)$. We have defined a new parameter called the matching dominating set and the matching domination number.We consider Kronecker product of two graphs, matching domination of product graphs and re-call the results associated with the matching domination of Kronecker product of graphs. We prove the following:*

In $G_1(k)G_2$ then $deg(u_i; v_j) = deg(u_i):deg(v_j)$.

If G_1 and G_2 are finite graphs without isolated vertices then $G_1(K)G_2$ is a finite graph without isolated vertices.

$jVG_1(k)G_2\ j = jVG_1\ jjVG_2\ j.\ jEG_1(k)G_2\ j =$
$2jEG_1\ jjVE_2\ j$

If G_1 and G_2 are regular graphs, then $G_1(K)G_2$ is also a regular graph. If G_1 or G_2 is a bipartite graph then $G_l(k)G_2$ is a bipartite graph.

The matching domination number of $c_4(k)K_m$ is 4.

If G_1; G_2 are two graphs without isolated vertices then $_{m}[G_1(k)G_2] = {}_{m}(G_1): {}_{m}(G_2)$ where $G_1(k)G_2$, is the Kronecker product of graphs.

***Keywords:** Kronecker Product of Graphs, Domination Set, Domination Number, Connected Graphs, Odd Cycles, Degree, Regular Graphs, Bipartite Graphs.*

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1 INTRODUCTION

Paul M Weichsel [11] defined the Kronecker product of graphs. He has proved a characterization for the product graphs to the connected graphs. He also obtained, if G_1 and G_2 are connected graphs with no odd cycles, then has exactly two connected components. E Sampathkumar [7] has proved that for a connected graph with no odd cycles $G_1(k)G_2 = 2G$ The concept of domination in graphs was first introduced by Ore[9].

2. KRONECKER PRODUCT OF GRAPHS

Definition 2.1

If G_1; G_2 are two simple graphs with their vertex sets as V_1 : fu_1; u_2::::::g and V_2 : fv_1; v_2::::::g respectively then the Kronecker product of these two graphs is defined to be a graph with its vertex set as V_1xV_2, where V_1xV_2 is the cartesian product of the sets V_1 and V_2 and two vertices (u_i; v_j), (u_k; v_l) are adjacent if and only if u_i; u_k and v_j; v_l are edged in G_1 and G_2, respectively. This product graph is denoted by G_1(k)G_2.
An illustration of the product graph of G_1(k)G_2 is given as follows.

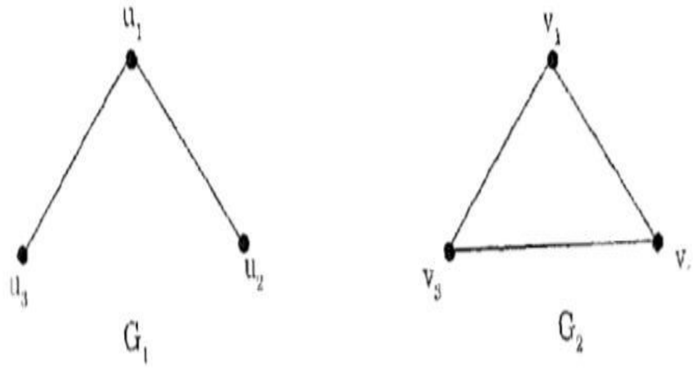

Figure 1.

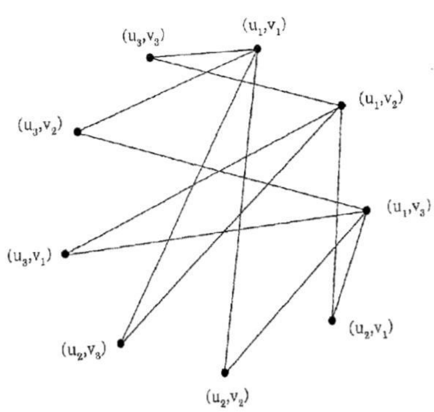

Figure 2: G_1 (k)G_2

Weichsel [11] has proved that if G_1; G_2 are connected graphs then G_1(k)G_2 is connected if and only if either G_1 or G_2 contains an odd cycle. It was further proved that if G_1; G_2 are connected graphs with no odd cycle then G_1(k)G_2 is a disconnected graph. Sampathkumar [32] has proved that, If G is connected graph with no odd cycles then G(k)K_2 = 2G.

Theorem 2.2
In G_1(k)G_2 then deg(u_i; v_j) = deg(u_i):deg(v_j).

Proof:
Suppose deg(u_i) = m and deg(v_j) = n.
i.e., u_i is adjacent with vertices u_1; u_2; :::::; u_m in G_1 and v_j is adjacent with vertices v_1; v_2; ::::::; v_m in G_2. Then in the product graph G_1(k)G_2, the vertex (u_i; v_j) is adjacent with following vertices.

$$\begin{array}{cccc}(u_1; v_1) & (u_1; v_2) & \cdots & (u_1; v_n)\\ (u_2; v_1) & (u_2; v_2) & \cdots & (u_2; v_n)\\ \vdots & \vdots & \vdots & \vdots \\ (u_m; v_1) & (u_m; v_2) & \cdots & (u_m; v_n)\end{array}$$

Also any other vertex $u_k; v_l$ in $G_1(K)G_2$ is not adjacent with $(u_i; v_j)$. If $k > m$ or $l > n$. For u_i is not adjacent with u_k if $k > m$ and v_j is not adjacent with v_l if $l > n$.
Hence $\deg(u_i; v_j) = \deg(u_i).\deg(v_j)$

Theorem 2.3
If G_1 and G_2 are finite graphs without isolated vertices then $G_1(K)G_2$ is a finite graph without isolated vertices.

Proof:
Since G_1 and G_2 are finite graphs, if follows that $G_1(K)G_2$ is also a finite graph by definition 2.1 since $G_1; G_2$ do not have isolated vertices.

$\deg_{G_1}(u_i) \neq 0$ for any i and so also $\deg_{G_1}(v_j) \neq 0$ for any j. Thus $\deg_{G_1(k)G_2}(u_i; v_j) \neq 0$ for any i and j (by Theorem 2.3). So $G_1(K)G_2$ do not have any isolated vertices.

It can be easily seen that, the number of vertices $G_1(K)G_2$ is the product of number of vertices in G_1 and G_2 and the number of edges in $G_1(K)G_2$ is twice the product of the number of edges in G_1 and G_2.

Theorem 2.4
(i) $|V_{G_1(k)G_2}| = |V_{G_1}||V_{G_2}|$

(ii) $|E_{G_1(k)G_2}| = 2|E_{G_1}||V_{E_2}|$

Proof:
It follows from the definition 2.1, $|V_{G_1(k)G_2}| = |V_{G_1}||V_{G_2}|$
we know that $|E_{G_1}| = e_1 = \frac{1}{2}\sum_{i \in V_1} d(u_j)$
and $|E_{G_2}| = e_2 = \frac{1}{2}\sum d(v_j)$

Now

$$|E_{G_1(k)G_2}| = \frac{1}{2}\sum_{i;j} d(u_i; v_j)$$

(By Theorem 2.2)

$$= \frac{1}{2}\{\sum_{i;j} d(u_i)d(v_j)\}$$

$$= \frac{1}{2}\{\sum_i d(u_i)\}\{\sum_j d(v_j)\}$$

$$= \frac{1}{2}\{2e_1\}\{2e_2\} = 2|E_{G_1}||V_{E_2}|$$

Theorem 2.5
If G_1 and G_2 are regular graphs, then G_1 (K) G_2 is also a regular graph.

Proof:
Suppose G_1 is a k_1 - regular graph and G_2 is a k_2 - regular graph then $deg(u_1) = k_1; 8u_1\ V_1$ and

$$deg(v_j) = k_2 8 v_j\ V_2$$

Let $(u_i; v_j)$ be any vertex in $G_1(k)G_2$ then (By Theorem 2.2)

$$deg(u_i; v_j) = deg(u_i):deg(v_j) = k_1k_2$$

Thus every vertex in $G_1(k)G_2$ is of degree k_1k_2 i.e., $G_1(k)G_2$ is k_1k_2 - regular.

Remark 2.6
However, it is to be noted that if G_1; G_2 are simple graphs then $G_1(k)G_2$ can never be a complete graph, for $(u_i; v_j)$ is not adjacent with $(u_i; v_k)$ for any j 6= k (By definition 2.1)

Theorem 2.7
If G_1 or G_2 is a bipartite graph then $G_l(k)G_2$ is a bipartite graph.

Proof:
Suppose G_1 is bipartite graph with bipartition (X,Y) where

$$X = fx_1; x_2; ::::::::::; x_m g$$

$$Y = fy_1; y_2; :::::::::; y_n g$$

Let $V_2 = fv_1; v_2; ::::::::::v_r g$
Then in $G_1(k)G_2$ the vertex set is

$$\begin{array}{cccc}
f(x_1; v_1) & (x_1; v_2) & ::: & (x_1; v_r) \\
(x_2; v_1) & (x_2; v_2) & ::: & (x_2; v_r) \\
. & . & . & . \\
. & . & . & . \\
. & . & . & . \\
(x_m; v_1) & (x_m; v_2) & ::: & (x_m; v_r) \\
(y_1; v_1) & (y_1; v_2) & ::: & (y_1; v_r) \\
(y_2; v_1) & (y_2; v_2) & ::: & (y_2; v_r) \\
. & . & . & . \\
. & . & . & . \\
. & . & . & . \\
(y_m; v_1) & (y_m; v_2) & ::: & (y_m; v_r)g
\end{array}$$

Now, no two vertices of the form $(x_i; v_j)$ and $(x_k; v_l)$ are adjacent since x_i and x_k are not adjacent. Similarly, no two vertices of the form $(y_i; v_j)$ and $(y_k; v_l)$ are adjacent since $y_i; y_k$ are not adjacent. Thus G_1 (k) G_2 is a bipartite graph with bipartition.
${}^XG_1(k)G_2$ and ${}^YG_1(k)G_2$
Where

$$X_{G_1(k)G_2} = f(x_i; v_j)=i = 1; 2; :::; m; j = 1; 2; ::::; rg$$

and

$$Y_{G_1(k)G_2} = f(y_i; v_j)=i = 1; 2; :::; n; j = 1; 2; ::::; rg$$

3. MATCHING DOMINATION NUMBER

Definition 3.1
A set S V is said to be a dominating set in a graph G if every vertex in V/S is adjacent to some vertex in S and the domination number $^{0\ 0}$ of G is defined to be the minimum cardinality of all dominating sets in G. We have introduced a new parameter called the matching domination set of a graph.

It is defined as follows:

Definition 3.2
A dominating set of a graph G is said to be matching dominating set if the induced subgraph < D > admits a perfect matching.

The cardinality of the smallest matching dominating set is called matching domination number and is denoted by $_m$
Illustration

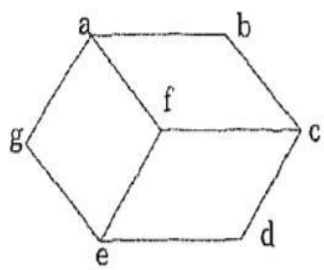

Figure 3:

In this graph {a, b, c, f, e, g } is a matching domination set, since this is a dominating set and the induced subgraph {a, b, c, e, f, g } has perfect matching formed by the edges of, bc, eg, {a,b,e,f } is also matching dominating set. Similarly, {a,b,c,g} is a matching dominating set where the induced subgraph of this set admits a perfect matching given by the edges be, ag.

However, there are no matching dominating sets of lower cardinality and it follows that the matching domination number of the graph in figure 3 is 4. Thus a graph can have many matching dominating sets of minimal cardinality. We make the following observations as an immediate consequence.

(a) Not all dominating sets are matching domination sets. For example in figure 3, {a,c,e } is a dominating set but it is not a matching dominating set.
(b) The cardinality of matching dominating set is always even. The matching dominating set D of a graph requires the admission of a perfect matching by the induced subgraph < D >. Thus it is necessary that D has an even number of vertices for admitting a perfect matching.

(c) Not all dominating sets with an even number of vertices are matching dominating sets. For example in figure 3,{b,d,g,f } is a dominating set containing an even number of vertices, but induced subgraph formed by these four vertices does not have a perfect matching.

(d) The necessary condition for a graph G to have to match dominating set is that G is a graph without isolated vertices. The matching domination number of the graph G (figure 3) is 4, whereas the domination number is 2 ; {a,d } being a minimal dominating set. If G is a graph with isolated vertices then any dominating set should include these isolated vertices and consequently, the induced subgraph of this set containing isolated vertices will not admit a perfect matching.

Theorem 3.3
The matching domination number of c_4(k)K_m is 4.

Proof :
Let V (C_4) = u_1; u_2; u_3; u_4 and V (K_m) = fv_1; v_2; ::::; v_mg

It can be easily seen that f< u_1; v_1 >; < u_2; v_2; < u_1; v_2 >; < v_2; v_1 >g will be a matching dominating set. The graph cannot have a matching dominating set of cardinality two, for if f< x; y >; < u; v >g is a matching dominating set of C_4 (k)K_m, it will not saturate the vertices < x; v >; < u; y >.

Hence the minimal matching domination number is 4 which is the same as the product of the matching domination number of C_4 and K_m.

$$= 2d^{4}_{4}\ e:2$$

$$= 4$$

Illustrate for the above theorem

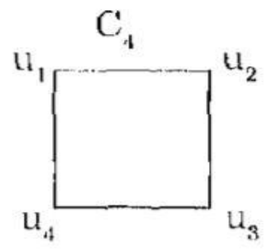

Figure 4: C_4

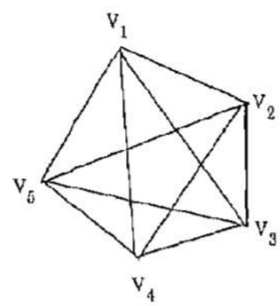

Figure 5: K5

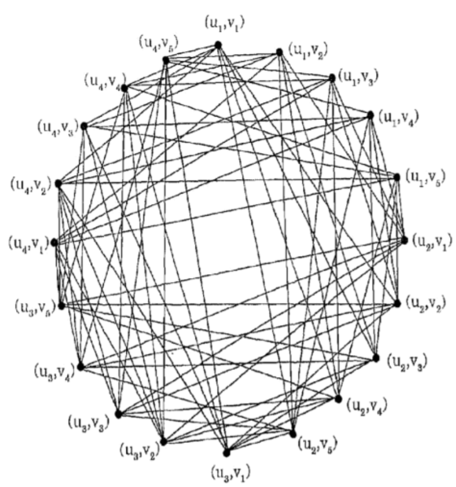

Figure 6: C_4(K)K_5

In general, we can prove that the matching domination number of product graph is equal to the product of matching domination numbers of the graphs.

Theorem 3.4
If G_1; G_2 are two graphs without isolated vertices then $_m[G_1(k)G_2]$ = $_m(G_1)$: $_m(G_2)$ where $G_1(k)G_2$, is the Kronecker product of graphs defined in 2.1.

Proof:
Let G_1 be a graph with p_1 vertices and G_2 be a graph with p_2 vertices. Let D_1 = fv_1; v_2; :::::::::::;
v_{2r}g
and D_2 = fw_1; w_2; :::::::::::; w_{2s}g
be minimal dominating sets of G_1 and G_2 respectively. Where f(v_1; v_2); (v_3; v_4); :::::::(v_{2r} $_1$; v_{2r})g and f(w_1; w_2); (w_3; w_4); :::::::(w_{2s} $_1$; w_{2s})g)) are pairs of adjacent vertices which constitute perfect matching in the induced subgraph < D_1 > and < D_2 > of G_1 and G_2 respectively.
Now consider Cartesian product of D_1 and D_2
D_1xD_2 = f(v_i; w_j)=1 i 2r; 1 j 2sg
Let (v; w) be any vertex of G_1(K)G_2. There exists a vertex (v_x; w_y) D_1xD_2 such that v is adjacent to v_x and w are adjacent to w_y. Thus D_1xD_2 is a dominating set of G_1(K)G_2. Deletion of any vertex in D_lxD_2 does not make D_1xD_2 a dominating set any more. For, if (v_i; w_j) is deleted from D_1xD_2 where v_i is adjacent to vertices fv_{i1}; v_{i2}; :::::::; v_{ix}g in G_1 and w_j is adjacent to the vertices fw_{j1}; w_{j2}; :::::::::; $w_{j\,y}$g in G_2, the vertices (v_{i1}; w_{j2}); (v_{i1}; w_{j2}):::::::(v_{ix}; $w_{j\,y}$) will not be adjacent to any of the vertices of D_1xD_2. Moreover, the minimality of D_1xD_2 can also be deduced from the following observation. "Suppose AxB is a matching dominating set of G_1(K)G_2 then A and B are matching dominating sets of G_1 and G_2 respectively". Consequently it follows that D_1xD_2 is a minimal dominating set of G_1(K)G_2. Further, it is also a matching dominating set for the vertices (v_1; w_2); (v_2; w_1); (v_1; w_1); (v_2; w_2); (v_1; w_3); (v_2; w_4); (v_1; w_4); (v_2; w_3); ::::::: forms a perfect matching in the induced subgraph < D_1xD_2 >. Thus D_1xD_2, is a matching dominating set of minimum cardinality.

Hence $_m[G_1(K)G_2]$ = $_m(G_1)$: $_m(G_2)$:

Illustration

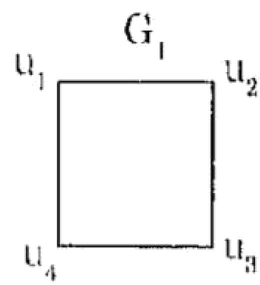

Figure 7: Matching dominating set: fu1; u_2g

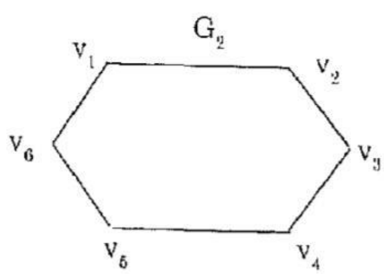

Figure 8: matching dominating Setfv1; v2; v4; v5g

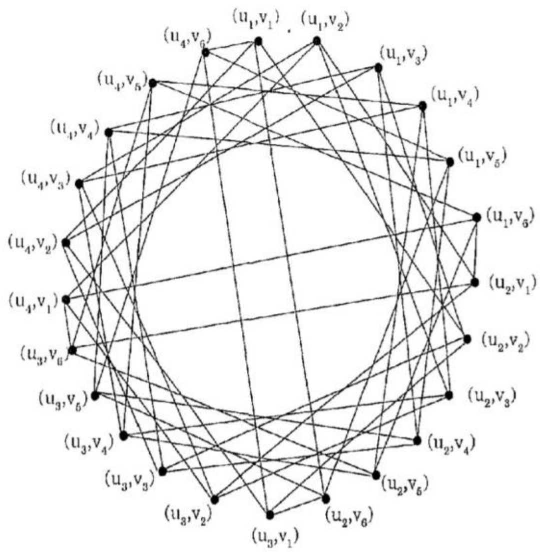

Figure 9: G_1(K)G_2

matching dominating set : f< u_1; v_1 >; < u_1; v_2 >; < u_1; v_4 >; < u_1; v_5 >; < u_2; v_1 >; < u_2; v_2 >; < u_2; v_4 >; < u_2; v_5 >g

= fu_1; $u_2$gxfv_1; v_2; v_4; v_5g

CONCLUSION

The study of Kronecker product graphs, the matching domination of Kronecker product graphs has been providing us sufficient stimulation for obtaining some in-depth knowledge of the various properties of the graphs. It is hoped that encouragement provided by this study of these product graphs will be a good straight point for further research.

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