Anti M-Fuzzy Subrings and its Lower Level M-Subrings

Nanthini .S. P.
Associate Professor, PG and Research Department of Mathematics, Hindusthan College of Arts and Science, Tamil Nadu.
nandhuran70@gmail.com

Munirathinam .C.
M.Phil. Scholar (FT), PG and Research Department of Mathematics, Hindusthan College of Arts and Science, Tamil Nadu.
muni18393@gmail.com

Abstract-- In this paper, we introduce the concept of an anti M-fuzzy Subring of an M-ring and lower level subset of an anti M-fuzzy Subring and discussed some of its properties.

Keywords- M-ring, fuzzy set, anti-fuzzy Subring, anti M-fuzzy Subring of an M-ring, level subset and level M-subrings.

I. INTRODUCTION


II. PRELIMINARIES

This section contains some definitions and results to be used in the sequel.

A) Anti fuzzy Subring:

Let R be a ring. A fuzzy subset A of R is called an anti-fuzzy subring of R, if ∀ x, y ∈ R.

1. A(x-y)≤A(x)υ A(y)
2. A(xy)≤A(x) υ A(y)

B) M-Ring:

Let R be a ring, M be a set if

1. mx∈R ∀ x∈R, m∈M.
2. m(xy) = m(x) + m(y) ∀x, y∈ R , m ∈ M.
3. m (xy) = (mx)y = x(my) ∀ x, y∈ R , m ∈ M. Then R is called an M-ring.
C) **M-Fuzzy Subring**: Let R be an M-ring and A be an anti fuzzy subring of R. Then A is called an anti M fuzzy Subring of R if

\[ \forall x \in R \text{ and } m \in M. \text{ Then, } A(mx) \leq A(x). \]

III. PROPERTIES OF ANTI M-FUZZY SUBRINGS

In this section, we discuss some of the properties of anti M-fuzzy Subring.

3.1 Theorem

Let S be an M-Subring of an M-ring R. Define a fuzzy subset A of R by

\[
A(x) = \begin{cases} 
  t_1 & \text{if } x \in S, \\
  t_2 & \text{otherwise}
\end{cases}
\quad \forall x \in R \text{ and } t_1, t_2, t_3 \in [0, 1].
\]

Then A is an anti M-fuzzy Subring of R.

**Proof:**

Assume that S be an M-Subring of an M-ring R.

**Case (i)**

If \( x, y \in S \) then \( xy \in S \)

\[ A(x) = t_1, \quad A(y) = t_1 \text{and } A(xy) = t_1 \]

Hence, \( A(xy) = t_1 = \max \{ A(x), A(y) \} \)

**Case (ii)**

If \( x \notin S \) (or) \( y \notin S \) then \( xy \notin S \)

\[ \therefore A(x) = t_2 \text{ (or) } A(y) = t_2 \text{and } A(xy) = t_2 \]

Hence, \( A(xy) = t_2 = \max \{ A(x), A(y) \} \)

**Case (iii)**

If \( x, y \in S \) then \( x-y \in S \)

\[ A(x) = t_1, \quad A(y) = t_1 \text{and } A(x-y) = t_1 \]

Hence, \( A(x-y) = t_1 = \max \{ A(x), A(y) \} \)

**Case (iv)**

If \( x \notin S \) (or) \( y \notin S \) then \( x-y \notin S \)

\[ \therefore A(x) = t_2 \text{ (or) } A(y) = t_2 \text{and } A(x-y) = t_2 \]

Hence, \( A(x-y) = t_2 = \max \{ A(x), A(y) \} \)

\[ \therefore A \text{ is an anti fuzzy Subring of } R. \]

Next we have to prove that A is an anti M-fuzzy Subring of R.

Since, S is an M-Subring of R.

We have \( mx \in S \forall m \in M \) and \( x \in S \)

\[ A(mx) = t_1 = A(x) \]

If \( x \notin S \). Then \( A(mx) = t_2 = A(x) \)
So, \( A(mx) \leq A(x) \)

Hence, \( A \) is an anti M-fuzzy Subring of \( R \).

### 3.2 Theorem

The union of any two anti M-fuzzy Subring \( S \) of an M-Ring \( R \) is always an anti M-fuzzy Subring of an M-ring \( R \).

**Proof:**

Let \( R \) be an M-Ring.

Assume that \( A \) and \( B \) be any two anti M-fuzzy Subring of \( R \).

**To prove:** \((A \cup B)\) is an anti M-fuzzy Subring of \( R \).

First we have to prove that \((A \cup B)\) is an anti fuzzy Subring of \( R \). Consider,

(i) \((A \cup B)(x-y) = A(x-y) \cup B(x-y) \leq \max\{A(x), A(y)\} \cup \max\{B(x), B(y)\} = \max\{A(x) \cup B(x), A(y) \cup B(y)\} \)

(ii) \((A \cup B)(xy) = A(xy) \cup B(xy) \leq \max\{A(x), A(y)\} \cup \max\{B(x), B(y)\} = \max\{A(x) \cup B(x), A(y) \cup B(y)\} \)

\((A \cup B)(m(x)) = \max\{A(m(x)), B(m(x))\} \leq \max\{A(x), B(x)\}\) as \( A \) and \( B \) are anti M-fuzzy Subring of \( R \). \( = (A \cup B)(x) \)

\((A \cup B)(m(x)) \leq (A \cup B)(x) \; \; \; \therefore (A \cup B) \) is an anti M-fuzzy Subring of \( R \).

### 3.3 Theorem

If \( A \) is an M-fuzzy subring of an M-ring \( R \) iff \( A^{c} \) is anti M-fuzzy subring of \( R \).

**Proof:**

Suppose \( A \) is a M-fuzzy subring of \( R \), \( \forall \; x, y \in R \) and \( m \in M \)

Consider,

\( A(x-y) \geq \{A(x) \wedge A(y)\} \)

\( A(x-y) \geq \min\{A(x), A(y)\} \)

\( \iff 1-\min\{1-A(x), 1-A(y)\} \)

\( \iff A^{c}(x-y) \leq 1- \min\{A^{c}(x), A^{c}(y)\} \)

\( \iff A^{c}(x-y) \leq \max\{A^{c}(x), A^{c}(y)\} \)

\( \iff A^{c}(x-y) \leq \{A^{c}(x) \vee A^{c}(y)\} \)

Next,

\( A(xy) \geq \{A(x) \wedge A(y)\} \)
\[ A(xy) \geq \min\{A(x), A(y)\} \]
\[ \iff 1 - A(xy) \leq 1 - \min\{1 - A(x), 1 - A(y)\} \]
\[ A^c(xy) \leq 1 - \min\{A^c(x), A^c(y)\} \]
\[ \iff A^c(xy) \leq \max\{A^c(x) \lor A^c(y)\} \]

Hence, \( A^c \) is anti-fuzzy subring of \( R \).

Next we have to prove \( A^c(mx) \leq A^c(x) \)

Consider,
\[ A(mx) \geq A(x) \]
\[ \iff 1 - A(mx) \leq 1 - A(x) \]
\[ \iff A^c(mx) \leq A^c(x) \]
\[ \therefore A^c \text{ is an anti } M\text{-fuzzy subring of } R. \]

### 3.4 Theorem

If \( A \) is an anti \( M\)-fuzzy subring of \( R \), then \( rAr^{-1} \) is also an anti \( M\)-fuzzy subring of \( R \) \( \forall r \in R \) and \( m \in M \).

**Proof:**

Assume that \( A \) is an anti \( M\)-fuzzy subring of \( R \).

First we have to prove that \( rAr^{-1} \) is an anti-fuzzy subring of \( R \).

**To prove:**

(i) \( rAr^{-1}(x-y) \leq rAr^{-1}(x) \lor rAr^{-1}(y) \)

(ii) \( rAr^{-1}(xy) \leq rAr^{-1}(x) \lor rAr^{-1}(y) \)

\[ \text{(i) } rAr^{-1}(x-y) = A(r^{-1}(x-y)r) \]
\[ \leq A\{r^{-1}xr \lor r^{-1}yr\} \]
\[ \leq A\{(r^{-1}xr) \lor A\{r^{-1}yr\}\} \]
\[ \leq \max\{A\{r^{-1}xr\}, A\{r^{-1}yr\}\} \]
\[ \leq \max\{rAr^{-1}(x), rAr^{-1}(y)\} \]
\[ rAr^{-1}(x-y) \leq rAr^{-1}(x) \lor rAr^{-1}(y) \]

\[ \text{(ii) } rAr^{-1}(xy) = A(r^{-1}(xy)r) \]
\[ \leq A\{r^{-1}xr \lor (r^{-1}yr)\} \]
\[ \leq A\{(r^{-1}xr) \lor A\{r^{-1}yr\}\} \]
\[ \leq \max\{A\{r^{-1}xr\}, A\{r^{-1}yr\}\} \]
\[ \leq \max\{rAr^{-1}(x), rAr^{-1}(y)\} \]
\[ rAr^{-1}(xy) \leq rAr^{-1}(x) \lor rAr^{-1}(y) \]

Hence, \( rAr^{-1} \) is also an anti fuzzy Subring of \( R \).

Next we have to prove that \( rAr^{-1} \) is an anti \( M\)-fuzzy Subring of \( R \).

Consider,
\[ rAr^{-1}(mx) = A(mr^{-1}xr) \]
\[ \leq A(r^{-1}Ar) \text{ as } A \text{ is anti } M\text{-fuzzy Subring} \]
\[ r \mathcal{A}^{-1}(x) = r \mathcal{A}^{-1}(mx) \leq r \mathcal{A}^{-1}(x) \]

Hence, \( r \mathcal{A}^{-1} \) is an anti \( M \)-fuzzy Subring of \( R \).

### 3.5 Theorem

The intersection of any two anti \( M \)-fuzzy subrings of an \( M \)-ring \( R \) is always an anti \( M \)-fuzzy Subring of an \( M \)-ring \( R \).

**Proof:**

Let \( R \) be an \( M \)-ring.
Assume that \( A \) and \( B \) be any two anti fuzzy subrings of \( R \).

**To prove:** \((A \cap B)\) is an anti \( M \)-fuzzy Subring of \( R \).

First we have to prove that \((A \cap B)\) is an anti fuzzy Subring of \( R \).

Consider,

(i) \[ (A \cap B)(x-y) = A(x-y) \cap B(x-y) \leq \max\{A(x), A(y)\} \cap \max\{B(x), B(y)\} = \max\{A(x) \cap B(x), A(y) \cap B(y)\} \]

\[ (A \cap B)(x-y) \leq \max\{(A \cap B)(x), (A \cap B)(y)\} \]

(ii) \[ (A \cap B)(xy) = A(xy) \cap B(xy) \leq \max\{A(x), A(y)\} \cap \max\{B(x), B(y)\} = \max\{A(x) \cap B(x), A(y) \cap B(y)\} \]

\[ (A \cap B)(xy) \leq \max\{(A \cap B)(x), (A \cap B)(y)\} \]

\[ \therefore (A \cap B) \text{ is an anti fuzzy Subring of } R. \]

Now, We have to prove that \((A \cap B)\) is an anti \( M \)-fuzzy Subring of \( R \). Consider,

\( (A \cap B)(m(xy))= \max\{A(m(xy)), B(m(xy))\} \)

\[ \leq \max\{A(xy), B(xy)\} \text{ as } A \text{ and } B \text{ are anti } M \text{-fuzzy Subring of } R. \]

\[ = (A \cap B)(xy) \]

\[ (A \cap B)(m(xy)) \leq (A \cap B)(xy) \]

\[ \therefore (A \cap B) \text{ is an anti } M \text{-fuzzy Subring of } R. \]

### IV. PROPERTIES OF LOWER LEVEL SUBSETS OF AN ANTI M-FUZZY SUBRING OF AN M-RING

In this section, we introduce the concept of lower level subset of an anti \( M \)-fuzzy Subring of an \( M \)-ring and discuss some of its properties.

**D) The lower level subset:** Let \( A \) be a fuzzy subset of \( S \). For \( t \in [0,1] \), the lower level subset of \( A \) is the set,

\[ A_t = \{ x \in S : A(x) \leq t \}. \]

### 4.1 Theorem

Let \( A \) be a fuzzy subset of an \( M \)-ring \( R \). If \( A \) is an anti \( M \)-fuzzy Subring of \( R \), then the lower level subsets \( A_t, t \in \text{Im}(A) \) are \( M \)-subrings of \( R \).
Proof:

Given that \( A \) is anti \( M \)-fuzzy Subring.

To prove: \( A_t \) is anti \( M \)-fuzzy Subring of \( R \).

First to prove \( A_t \) is an anti fuzzy Subring of \( R \).

Let \( t \in \text{Im} (A) \) and \( x, y \in A_t \).

We have \( A(x) \leq t, A(y) \leq t \).

Now,

\[
A(x-y) \leq \max \{ A(x), A(y) \} \\
\leq \max \{ t, t \} \\
\leq t \\
A(x-y) \leq t \\
\text{ie.,} x-y \in A_t \\
\]

Moreover, if \( x, y \in A_t \).

We have,

\[
A(x) \leq t, A(y) \leq t \\
A(xy) \leq \max \{ A(x), A(y) \} \leq \max \{ t, t \} \\
A(xy) \leq t \\
\text{ie.,} xy \in A_t \\
\]

Hence, \( A_t \) is an anti fuzzy subring of \( R \).

Now, to prove \( A_t \) is an anti \( M \)-fuzzy subring of \( R \).

For any \( x \in A_t \) and \( m \in M \).

\[
A(mx) \leq A(x) \\
\leq t \\
\]

Hence, \( A(mx) \leq t \)

\[
\text{ie.,} mx \in A_t \\
\]

Hence,

\( \therefore A_t \) is an anti \( M \)-fuzzy subring of \( R \).
4.2 Theorem

Any M-subring $H$ of an M-ring $G$ can be realized as a lower level M-subring of some anti M-fuzzy subring of $R$.

Proof:
Let $A$ be a fuzzy subset and $x \in R$.

Define,

$$A(x) = \begin{cases} 0 & x \in S \\ t & x \notin S, \text{ where } t \in 0,1 \end{cases}$$

Prove that $A$ is an anti M-fuzzy subring of $R$.

First to prove $A$ is an anti fuzzy subring of $R$.

Case (i)
Suppose $x, y \in S$.
Then $x \in S, y \in S$ and $x - y \in S$

$$A(x) = 0, A(y) = 0 \text{ and } A(x - y) = 0$$

Hence,

$$A(x - y) \leq \max\{A(x), A(y)\}$$

Case (ii)
Suppose $x \in S$ and $y \notin S$
Then $x - y \notin S$

$$A(x) = 0, A(y) = t \text{ and } A(xy^{-1}) = t$$

Hence,

$$(x - y) \leq \max\{A(x), A(y)\}$$

Case (iii)
Suppose $x, y \in S$.
Then $x \in S, y \in S$ and $xy \in S$

$$A(x) = 0, A(y) = 0 \text{ and } A(xy) = 0$$

Hence,

$$A(xy) \leq \max\{A(x), A(y)\}$$

Case (iv)
Suppose $x \in S$ and $y \notin S$
Then $xy \notin S$

$$A(x) = 0, A(y) = t \text{ and } A(xy) = t$$
Hence,
\[ A(xy) \leq \max\{A(x), A(y)\} \]
\[ \therefore A \text{ is an anti fuzzy subring of } R. \]
Now, \( \forall m \in M \) and \( x \in S \) then \( mx \in S \)
\[ \therefore A(x) = 0 \text{ and } A(mx) = 0 \]

Hence,
\[ A(xy) \leq A(x) \]
Now, \( \forall m \in M \) and \( x \in S \), then \( mx \in S \) (or) \( mx \in S \)
\[ \therefore A(x) = t \text{ and } A(mx) = 0 \text{ (or) } t. \]

Hence,
\[ A(mx) \leq A(x) \]
\[ \therefore A \text{ is an anti M-fuzzy subring of } R. \]

For this anti M-fuzzy subring,
\[ = S \]

CONCLUSION
In this paper, we define a new algebraic structure of anti M-fuzzy Subring of an M-ring and lower level subset of an anti M-fuzzy subring and studied some of its properties. Further, we wish to define the anti M-fuzzy normal Subring of an M-ring and its lower level subsets and also the same in Intuitionist fuzzy and other some rings are in progress.

ACKNOWLEDGEMENT
The authors would like to thank the referee for the valuable comments which helped us to improve this manuscript

REFERENCE