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Rogers -Ramanujan Identities with Golden Ratio

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Abstract: - Ramanujan was an eminent Indian Mathematician of 20th century. During the years 1903-1914 he discovered most of the mathematical results. In this note, we focus on the results associated with Rogers- Ramanujan identities of first and second kind and Golden Ratio.

Key words: Golden ratio, Partition, Rogers-Ramanujan identities.

I. Introduction

Srinivasa Aiyangar Ramanujan was an eminent Indian Mathematician of 20th century. During the years 1903-1914 he discovered most of the mathematical results .He made contribution to analytical theory of numbers and worked on elliptic functions, continued fractions, and infinite series. In this note, we focus on the Rogers- Ramanujan identities of first and second kind and its association with Golden Ratio.

II. Preliminaries

We define the partition function, generating function and continued fraction. This leads to Ramanujan's continued fraction and Roger's - Ramanujan identities.

Definition 1: Partition function: A partition function p(n) represents the number of possible partitions of a natural number n. A partition of a positive integer n is a non-increasing sequence of positive integers whose sum is n.

The first few values are p(0) = 1 , p(1)=1, p(2)=2, p(3)=3, p(4)=5, p(5)=7,..... and p(n) = 0 for n negative.

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right)$$

Definition 2: Generating Functions -The partition function has a nice generating function in q.

$$\sum_{n=1}^{\infty} p(n)\,)\, q^n = \prod \frac{1}{(1-q^n)} = \frac{1}{(1-q)} \cdot \frac{1}{(1-q^2)} \cdot \frac{1}{(1-q^3)} \cdot \frac{1}{(1-q^4)} \cdots\cdots\cdots \text{where } p(0) = 1$$

The Rogers -Ramanujan identities are the reformulation in terms of partition function. They were first discovered by Rogers in 1894 and rediscovered by Ramanujan in 1913.

For $|q| < 1$,

$\sum q^{n2}/ (q)_n = 1 / [(q ; q^5)_{\infty} (q^4 ; q^5)\infty]$ for $n \geq 0$, **Rogers- Ramanujan Identity of First Kind which can be written as,**

$$1 + \frac{q}{(1-q)} + \frac{q^4}{(1-q)(1-q^2)} + \frac{q^9}{(1-q)(1-q^2)!1-q^3)} + \ldots\ldots\ldots$$

$$= \frac{1}{(1-q)(1-q^6)(1-q^{11})(1-q^{16})} \ldots\ldots \times \frac{1}{(1-q^4)(1-q^9)(1-q^{14})\ldots\ldots} \qquad \ldots\ldots. (i)$$

$\sum q^{n2 + n} / (q)_n = 1 /[(q^2 ; q^5)_{\infty} . (q^3 ; q^5)\infty]$ for $n \geq 0$, **Rogers- Ramanujan Identity of Second Kind which can be written as**,

$$1 + \frac{q^2}{(1-q)} + \frac{q^6}{(1-q)(1-q^2)} + \frac{q^{11}}{(1-q)(1-q^2)!1-q^3)} + \frac{q^{18}}{(1-q)(1-q^2)(1-q^3)(1-q^4)} + \ldots\ldots\ldots$$

$$= \frac{1}{(1-q^2} \frac{1}{)(1-q^7)(1-q^{12})(1-q^{17})} \ldots\ldots \times \frac{1}{(1-q^2)(1-q^{13})(1-q^{18})\ldots\ldots} \qquad \ldots\ldots\ldots. (ii)$$

These identities are equivalent forms of

i) The number of partitions of n into parts, any two of which are differ by at least 2, equals the number of partitions of n into parts congruent to $\pm$ mod 5.

ii) The number of partitions of n into parts > 1, any two of which differ by at least 2, equals the number of partitions of n into parts congruent to $\pm$ mod 5.

Partition identities are related to representation theory, modular forms, statistical mechanics, etc.

Theses identities can be written as follows.

$$\mathbf{G (q)} = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1-q)(1-q^2)(1-q^3)\ldots\ldots(1-q^n)} = \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})}$$

$$\mathbf{H (q)} = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(1-q)(1-q^2)(1-q^3)\ldots\ldots(1-q^n)} = \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+3})(1-q^{5n+3})}$$

Remark 1: Roger's -Ramanujan identities are Mock Theta Functions of order 5.

Consider $$\frac{H(q)}{G(q)} = \cfrac{1}{1+\cfrac{q}{1+\cfrac{q^2}{1+\cfrac{q^3}{1+\cfrac{q^4}{1+\cdots\ldots}}}}}$$

If we put q=1, we get $$\frac{H(1)}{G(1)} = \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cdots\ldots}}}}}$$

If we break this as 1,1+1 , $1+\frac{1}{2}$, $1+\frac{1}{\frac{3}{2}} = \frac{5}{3}$, ...

i.e. 1, 2, 3/2, 5/3, 8/5,.......the nth term u_n satisfies the recurrence relation $u_n = u_{n-1} + u_{n-2}$

and $u_n = \frac{Fn}{F_{n-1}}$, where F_n are the nth **Fibonacci Numbers.**

This sequence is convergent as n→∞, called as Fibonacci sequence.

Rogers-Ramanujan continued fraction R (q) is defined as for $|q| < 1$, we have

$$R(q) = \cfrac{q^{1/5}}{1+\cfrac{q}{1+\cfrac{q^2}{1+\cfrac{q^3}{1+\cfrac{q^4}{1+\cdots}}}}}$$

For q=1, we get $R(1) = \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cdots\ldots}}}}}$ which is the Golden ratio $\beta = \frac{1+\sqrt{5}}{2}$ and $\alpha = \frac{1-\sqrt{5}}{2}$

Remark 2: Thus q=1 in Ramanujan continued fraction gives the Golden ratio α and β, we discuss some of the properties of Golden ratio.

Properties of Golden ratio i) $\alpha\beta = -1$

ii) $\alpha + \beta = 1$

iii) $\alpha = -\frac{1}{\beta}$

iv) $\alpha^2 = \alpha(1-\beta) = \alpha - \alpha\beta = \alpha - (-1) = \alpha + 1$

v) $\alpha^n = \alpha F_n + F_{n-1}$ where $F_n = \alpha^n - \beta^n / \alpha - \beta$, $n \geq 0$, the nth Fibonacci Number.

vi) $\beta^n = \beta F_n + F_{n-1}$, $n \geq 0$

Results Associated with Rogers-Ramanujan identities and Golden Ratio

These results were proved by Hei- Chi- Chan in a book named as, An Invitation to q-series

1) $R(q) = q^{1/5}\frac{H(q)}{G(q)}$

2) $R(e^{-2\pi}) = \sqrt{2+\beta} - \beta$

3) $G(q) = \frac{f(-q^{1/5})}{\sqrt{5}\, f(-q)} (\beta J(\beta) - \alpha J(\alpha))$

4) $q^{1/5} H(q) = \frac{f(-q^{1/5})}{\sqrt{5}\, f(-q)} (J(\beta) - J(\alpha))$

5) $G(q) = \frac{1}{\sqrt{5}} \left(\frac{\beta}{J(\alpha)} - \frac{\alpha}{J(\beta)} \right)$

6) $H(q) = \frac{1}{q^{1/5}} \frac{1}{\sqrt{5}} \left(\frac{1}{J(\alpha)} - \frac{1}{J(\beta)} \right)$

7) $\sum_{n\geq 0} \frac{q^{n^2}}{(q)_n} = \frac{1}{\sqrt{5}} \left(\beta \prod_{n=1}^{\infty} \frac{1}{(1+\alpha q^{1/5}+q^{2n/5})} - \alpha \prod_{n=1}^{\infty} \frac{1}{(1+\beta q^{1/5}+q^{2n/5})} \right)$

8) $\sum_{n\geq 0} \frac{q^{n^2+n}}{(q)_n} = \frac{1}{q^{1/5}} \frac{1}{\sqrt{5}} \left(\prod_{n=1}^{\infty} \frac{1}{(1+\alpha q^{1/5}+q^{2n/5})} - \prod_{n=1}^{\infty} \frac{1}{(1+\beta q^{1/5}+q^{2n/5})} \right)$

9) $q^{1/10}\sqrt{\frac{5f\,(-q^5)R(q)}{f(-q)}} = \left(\prod_{n=1}^{\infty} \frac{1}{(1+\alpha q^{1/5}+q^{2n/5})} - \prod_{n=1}^{\infty} \frac{1}{(1+\beta q^{1/5}+q^{2n/5})} \right)$

10) $q^{1/10}\sqrt{\frac{5f\,(-q^5)}{f(-q)R(q)}} = \left(\prod_{n=1}^{\infty} \frac{1}{(1+\alpha q^{1/5}+q^{2n/5})} - \prod_{n=1}^{\infty} \frac{1}{(1+\beta q^{1/5}+q^{2n/5})} \right)$

Remark 2: In Roger's- Ramanujan identities we observe Equality between the sums and products are expressed by these identities.

Remark 3: The sums are on the right hand side which are the generating functions and products are on the left hand side are congruence relations.

Conclusion

We have defined the q-series, which lead to partition function. Using partition function continued fraction is defined. Ramanujan's continued fraction at q=1 give rise to Golden ratio. Also we have seen that from Ramanujan's continued fraction Roger's -Ramanujan identities are derived. Finally, we give results which are associated with Rogers’s’s-Ramanujan identities and Golden ratio.

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