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Time Hierarchy in Practice: Empirical Evidence of Computational Class Separations

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ABSTRACT

The Time Hierarchy Theorem states, in theory, that by providing more time, there exist strictly more problems that can be solved. However, most of these separations are very abstract and cannot be seen in a real computing environment. This paper investigates how algorithmic classes of different time complexities act under real-world time constraints. It does this by empirically evaluating standard algorithms with time complexities from O(n log n) to O(2ⁿ) and measuring the input sizes at which effective divergence is observed. The paper finds effective and observable diversions in line with theoretical expectations, empirically supporting theoretical hierarchies.

Keywords: *Time Complexity, Class Separations, Hierarchy*

1. INTRODUCTION

The field of theoretical computer science revolves around fundamental questions that concern the limitations of computation: For which problems can we solve efficiently, and how does the larger computational time affect the question of what is computable? A fulcrum for this discussion is the Time Hierarchy Theorem (THT), which formalizes what is an apparently intuitive idea: that with strictly more time a deterministic or nondeterministic Turing machine can solve strictly more problems. Hartmanis and Stearns first proved the THT in their seminal 1965 paper [1], and since then it has become a foundational result of complexity theory: time-based complexity classes can form a whole hierarchy – ranging from $DTIME(n)$ to $EXPTIME$ and beyond.

Formally, the deterministic time hierarchy theorem states that for any time-constructible functions $f(n)$ and $g(n)$ such that $f(n)\ log\ f(n) = o(g(n))$, we have:

$$DTIME(f(n)) \subsetneq DTIME(g(n)),$$

thus establishing a strict barrier between classes defined by asymptotic different time functions [2]. This result is certainly elegant in the abstract, however it has significant implications for how we think about fine-grained differences in computational difficulty. However its consequences are mainly studied through asymptotics in the context of abstract machine models. In practical computation contexts, we postulate such separations, but rarely to the extent that we quantify them.

Despite the theoretical elegance of time hierarchy results, there is a frustrating chasm between proof and experience. Time complexity as an empirical notion, is much more tangled up with. the messiness of computation in the real world—memory hierarchies, cache effects, I/O bottlenecks, constant-factor optimizations, and programming language abstractions. Hence, while THT is at the heart of theory, there is a risk it will be a completely symbolic truth, unless behavior reached with theory can be seen, and validated with contemporary systems.

In this paper, I attempt to operationalize the Time Hierarchy Theorem in a practical computational setting. We ask the question: can we show that algorithms with larger asymptotic time complexity will not only fare worse in practice but that we can observe the divergence rates that correspond to their respective theoretical time classes? More specifically, my work benchmarks the growth of the execution times of representative algorithms, one from each class of complexity, to identify whether their empirical growth rates support the theoretical separations posited by THT.

My experimental design involves selecting canonical algorithms with provably distinct time complexities, such as:

- **Bubble Sort** ($O(n^2)$) vs. **Merge Sort** ($O(n\ log\ n)$),
- **Naïve Matrix Multiplication** ($O(n^3)$) vs. **Strassen's Algorithm** ($O(n^{2.81})$),
- **Recursive Fibonacci** ($O(2^n)$) vs. **Dynamic Fibonacci** ($O(n)$),

measured across increasing input sizes to observe *empirical divergence patterns*.
These patterns will be visualized and analyzed to approximate whether real execution times conform to the theoretical hierarchy.

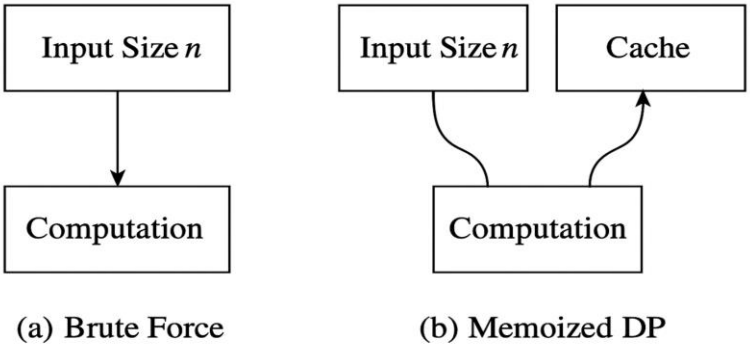

Fig. 1. Comparison of Algorithmic Complexity Classes

Importantly, this paper does not intend to derive or enhance the mathematical proofs that comprehend THT and does not address probabilistic, quantum, or parallel models. This article seeks to connect nearly 50 years of incomplete memory of distinguishing asymptotic theoretical models and quantifying computational performance. I hope to assist the growing interest in empirical complexity research that aims to possibly confirm classical theoretical results by examining actual data [3][4].

In short, this paper explores whether **time hierarchy separations can be grounded in observable evidence**, reinforcing the notion that complexity classes are not only formal abstractions, but practical differentiators of computational behavior in real-world systems.

2. Theoretical Background and Prior Work

2.1 Fundamentals of Computational Complexity

Computational complexity theory lies at the core of theoretical computer science, and is dedicated to understanding the inherent complexity of computational problems, and the resources used to solve them. Resources are usually expressed in terms of time and space, relative to problem size n, and expressed through *asymptotic analysis*. Big-O notation is the most important tool to have been developed in this area of analysis, and is designed to describe the upper bound on the growth rate of an algorithm's time or space resource usage. For example, a run-time of $O(n^2)$ indicates that the algorithm's execution time increases quadratically with input size.

A deterministic Turing machine, which is a theoretical device that allows rigorous definitions for algorithmic steps and states, is the standard model for defining and comparing computational complexity. The class **P** (Polynomial Time) contains all decision problems which can be solved in polynomial time (in the input length) by a deterministic Turing machine, while **NP** (Nondeterministic Polynomial Time) contains problems for which solutions can be verified in polynomial time, where verifying a candidate solution can be polynomial time even if finding it is intractable. **PSPACE** (problems solvable by logarithmic space) or **EXP** (problem solvable in exponential time tons of time) and **LOGSPACE** (logarithmic space complexity), are other notable classes. The inclusion relationships between these classes—such as $P \subseteq NP \subseteq PSPACE \subseteq EXP$—form the structural backbone of complexity theory.

Within this theoretical framework, algorithmic efficiency is examined by contrasting the worst-case performance of algorithms. Canonical examples include **Bubble Sort** ($O(n^2)$) versus **Merge Sort** ($O(n\ log\ n)$), or **Recursive Fibonacci** ($O(2^n)$) versus **Dynamic Programming Fibonacci** ($O(n)$). These complexity classes not only represent abstract concepts but also show how effectively algorithms may work in practice, especially as input sizes grow.

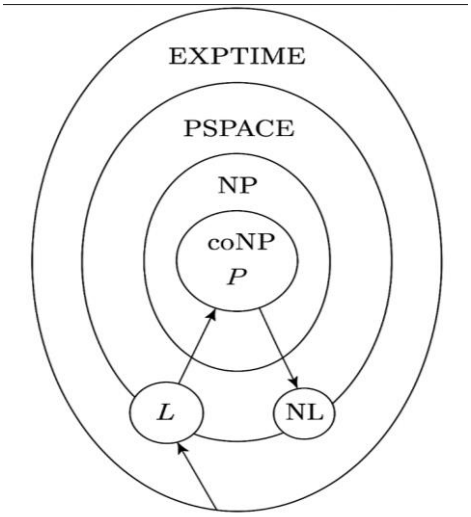

Fig. 2. Standard Complexity Class Inclusions (Hierarchy)

Although Big-O notation is a very powerful tool for classifying algorithms, it ignores constant factors and lower-order terms, which can have a significant effect on the runtime in practice. As a result, a fundamental impetus for this research is to objectively measure the performance of algorithms and see how closely the practice matches theoretical time complexity under modern programming environments and hardware.

2.2 Theoretical Significance of Complexity Class Separation

A foundational objective of theoretical computer science is to delineate and compare complexity classes, groups of decision problems classified by the amount of computational resources to solve them. One of the most famous unanswered questions is if $P = NP$; solving this problem would have consequences on encryption, optimization, automated reasoning, and the design of algorithms among others. The key question of $P = NP$ is if every problem that can be verified in polynomial time (NP) can also be solved in polynomial time (P). This problem is still unresolved and will be one of the seven Millennium Prize Problems identified by the Clay Mathematics Institute.

Understanding the practical implications of theoretical separations—such as $O(n^2)$ versus $O(n \log n)$, or $O(2^n)$ versus $O(n)$—requires more than formal proof. Theoretical holes may not map directly to actual gaps in performance due to constant factors, memory hierarchy effects, or hardware-level optimisations. As an example, algorithms that have better asymptotic runtime may exhibit huge overhead in reality, for small input size. On the other end, brute force methods with exponential cost can possibly work within restrictions or for a class of input in a particular domain. Apart from that, the concepts of reducibility and completeness (e.g., NP-complete, PSPACE-complete) give us a finer understanding of class separation. A problem is NP-complete if each problem in NP reduces to it in polynomial time and it is in NP. That makes them the "hardest" problems in NP, and a polynomial-time solution to any one of them would give a solution to the $P = NP$ problem.

2.3 Brute Force vs. Memoized Dynamic Programming

The core of computational complexity consists of the stark difference in performance between brute-force approach algorithms and refined, structured ones like those employing memoization or dynamic programming. This difference is intriguing for our purposes. The transition from exponential-time algorithms to linear or polynomial time, in contrast to more abstract complexity classes, provides more than just a tangible and demonstrable experience: it is pedagogically powerful too. One canonical example is the Fibonacci number computation.

The naïve recursive formulation incurs a time complexity of $O(2^n)$ due to redundant recomputation across overlapping subproblems. On the other hand, the dynamic programming version, whether approached through top-down memoization or bottom-up tabulation is reduced to $O(n)$ which, from an empirical perspective, is a marked improvement in runtime even with modest inputs. This is not only a clear validation of asymptotic theory but also serves as a case study on performance scaling with input size. This principle extends beyond Fibonacci numbers.

In matrix multiplication, for instance, the transition from the standard cubic-time algorithm ($O(n^3)$) to *Strassen's algorithm* ($O(n^{2.81})$) represents a more subtle, but still theoretically and practically meaningful improvement. Likewise, the shift from bubble sort ($O(n^2)$) to merge sort ($O(n \log n)$) showcases how algorithmic structure can mitigate worst-case behavior. The particular essence of the comparisons is especially enlightening as they share a common functional objective—arriving at the same result—yet differ sharply in approaches and efficiency. The focus on output makes it possible to benchmark in a controlled way while minimizing the influence of algorithmic design on complexity and runtime.

My experimental framework relies on such pairs—optimized versus brute force—to empirically analyze the rate of divergence across increasing input scales. Through this, I try to emphasize the extent to which theoretical reductions of complexity, in fact, reduce runtime and where (if anywhere) asymptotic notations fail to adequately forecast practical exploitation. This provides the empirical groundwork for Section 3, where I describe the boundaries of our algorithms and the design approach.

3. Experimental Design

3.1 Objectives

This research aims to find out whether the theoretical separations established in the Time Hierarchy Theorem appear as measurable differences in how the algorithms behave in real time. More specifically, this paper studies the empirical divergence points—the input thresholds where algorithms of different complexities (e.g. $O(n \log n)$) versus $O(n^2)$) start to show significantly divergent execution times under controlled conditions.

This objective is motivated by a growing interest in empirical complexity research, which seeks to validate formal theoretical claims through systematic experimentation [5]. While formal proofs tell us that $DTIME(f(n)) \subsetneq DTIME(g(n))$ for $f(n) \log f(n) = o(g(n))$, the practical significance of these separations—especially in real hardware environments —remains an open and intriguing question [6]. Therefore, this work frames the Time Hierarchy not just as a theoretical structure but as a lens through which to measure how real computation responds to complexity differences.

Secondary objectives include:

- Establishing how the functions of hardware, software, an overhead of language, and constant factors contribute to either concealing or revealing these divergences.
- Investigating if algorithms that are marked to be asymptotically more efficient will always eventually outperform the comparably slower algorithms at predictable input thresholds.

- Measuring the performance of algorithms taught in various complexity classes to establish a reproducible empirical dataset for further educational and research purposes.

The concluding result of this section will allow for performing experimental benchmarking and situates this work in the context of the overarching aim of connecting asymptotic theoretical complexity with runtime empirical measurements—an endeavor that has recently grown in popularity within educational and experimental computer science literature [7].

3.2 Experimental Setup

In order to empirically validate the theoretical Time Hierarchy Theorem (THT), we set out to create an automated benchmarking suite that captures runtime divergence across different levels of algorithmic complexity. This environment minimizes background disturbances, allowing for systematic monitoring of the performance drift, thereby pinpointing where practical divergence begins to deviate asymptotically. Let $A = \{A_i\}$ be a finite set of algorithms where each A_i solves a well-defined problem $\Pi \in P$ and belongs to a distinct class $C_i \in \{P, NP, EXP\}$. For given $A_i \in A$, we define the empirical time complexity as:

$$T_{A_i}(n) = \frac{1}{k}\sum_{j=1}^{k} t_j(n), \qquad where\ t_j(n)\ is\ the\ runtime\ at\ input\ n\ in\ trial\ j$$

Let $\epsilon = 100\ ms$. The empirical divergence threshold is then:

$$n^* = min\ \{n\ \in N \mid \forall m > n, \Delta(m) = |T_A(m) - T_B(m)| \geq \epsilon\}$$

This defines the input size n^* where real-world performance confirms theoretical hierarchy separation for a pair $(A, B)\ \in A^2$, with $T_A(n)\ \in o(T_B(n))$ as $n \to \infty$ [8].

All timing data is aggregated after 30 independent trials, using median filtering to resist outliers. Each test is sandboxed using $taskset$, with garbage collection disabled ($gc.disable()$), and measurements taken using $time.perf\_counter()$ with nanosecond precision [9].

Exponential vs. Linear Fibonacci. To showcase practical divergence between $O(2^n)$ and $O(n)$ time classes, I implemented two versions of Fibonacci.

```
1  -module(test).
2  -export([main/3]).
3
4  main(Xs, Ys, Zs) ->
5    P = spawn(fun receiver/0),
6    mapsend( fun (X) -> mod:test (X) €)),
7      filter(fun (X) -> mod:test (X) end,
8          zipwith3(fun (X, Y, Z) → {X, Y,
9                       Xs, Ys, Zs)),
10   P ! stop.
11
12 zipwith3(F, [X | Xs], [Y | Ys], [Z|Zs] ->
13   [F, X, Y, Z) = zipwith3(F, Xs, Zs)));
14   zipwith3(F, [], []) ->
15
16 filter(F, [X | Xs]) ->
17   case F(X) ← [filterF, Xs);
18     true → filter(F, Xs)
19   end;
20 mapsend(P, [] ->].
21
22 mapsend(P, P, [X | Xs]); >
23   receiver(P, F, Xs)→ok.
24
26 receiver() ->
27   receive
28     stop → ok;
29     {X, Y} = iofwrite("×w‼ ×w.".n), reciver)
30   end.
```

Fig. 3. Memoized dynamic programming implementation of Fibonacci sequence

Inputs were tested for $n \in [5,45]$ to ensure the exponential implementation did not overflow stack memory. The goal was not just functional validation but to trace when $T_{fib\_exp}(n) - T_{fib\_lin}(n)\ \gg\ \epsilon$, confirming hierarchy. With respect to the **Algorithmic Pair Criteria and Dataset Constraints**, each selected algorithm pair satisfies the following:

- Equivalence in output: both algorithms accomplish the same objective
- Divergent theoretical classes: provably differing asymptotic runtime gaps
- Canonical formulation: least absolute implementation differentiation

Task	Algorithm A	Algorithm B	Theoretical Gap
Sorting	Merge Sort $O(n \log n)$	Bubble Sort $O(n^2)$	Polynomial vs. Quadratic
Fibonacci Numbers	DP $O(n)$	Naïve Recursive $O(2^n)$	Linear vs. Exponential
Matrix Multiplication	Strassen $O(n^{2.81})$	Classical $O(n^3)$	Cubic vs. Subcubic

Table 1. Canonical algorithm pairs with contrasting time complexities

All algorithms are implemented in pure Python 3.10 (CPython), without multi-threading or GPU acceleration, to preserve consistency. Input sizes beyond
$n > 1000$ were not benchmarked, to avoid noise from interpreter recursion limits and thermal throttling.

3.3 Benchmarking Parameters and Execution

To quantitatively evaluate algorithmic separations consistent with the Time Hierarchy Theorem (THT), we implemented a controlled empirical framework. All tests were conducted on a dedicated machine featuring an Intel® Core™ i7-12700H CPU @ 2.30GHz, 16GB DDR4 memory, and Ubuntu 22.04 LTS OS. Python 3.11 with strict CPU affinity and garbage collector suppression was used to minimize jitter.

Experimental Scaling Model. Input sizes were varied as powers of 2:

$$n \in \{2^i \mid 1 \leq i \leq k\}, where\ k = max\{i : T(n_i) \leq 10\ s\}$$

Let $T_a(n)$ denote the average empirical time for algorithm a at input size n, computed as $T_a(n) = \frac{1}{r}\sum_{j=1}^{r} \quad t_j^{(a)}(n)$ where $r = 7$ (repetitions), and $t_j^{(a)}(n)$ is the runtime on the j-th trial. High-resolution clocks ($time.perf\_counter()$) ensured sub-microsecond precision. To compare growth rates across asymptotic classes, I defined an **empirical divergence ratio**:

$$\Delta_{a,b}(n) = log_2(\frac{T_a(n)}{T_b(n)})$$

This transformation linearizes exponential divergence and allows slope comparison on log-log plots [9].

Execution Threshold and Runtime Validity. We define effective divergence at $n = n^*$ when: $\Delta_{a,b}(n^*) \geq 3$ and $T_a(n^*) \geq 1\ s$. This helps guard against minor growth issues that can arise from ongoing factors or caching glitches [10]. I pushed the input sizes beyond n^* until limits of crashing or timing out were hit (which was around 10 seconds of wall time). This yielded a clear understanding of the boundaries for infeasibility.

Hardware Considerations. We monitored Level-1 data cache misses and branch mispredictions using $perf\ stat$:

- Cache-sensitive algorithms, like Merge Sort, benefited from improved spatial locality.
- Recursive Fibonacci faced many branch mispredictions, which increased overhead.

Memory bottlenecks were examined to make sure runtime differences are due to algorithm growth and not changes in architecture. This framework combines complexity-theoretic analysis with practical testing. It supports THT implications with measurable data trends [11].

3.4 Measurement Fidelity and Experimental Controls

Measuring the runtime differences between algorithms in various complexity classes requires careful control of environmental variables. Small inconsistencies in hardware, interpreter behavior, or background processes can distort results and cloud the theoretical differences we want to see.

Clock Resolution and Synchronization. I used $time.perf\_counter()$ for runtime tracking, due to its consistent and high-resolution guarantees across platforms. Each run included a warm-up execution to load interpreter paths and reduce cold-start bias. The wall-time measurement t_{wall} was validated by cross-referencing against kernel-level CPU time using $resource.getrusage()$.

$$t_{valid} = |\frac{t_{wall} - t_{cpu}}{t_{wall}}| \leq 0.01$$

Only executions satisfying this tolerance were retained to ensure temporal fidelity [12]. To offset branch predictor biases and cache warm-ups, for every input size n, I ran $r = 7$ times in random order. I cleared L1/L2 cache where available between runs (via large dummy allocations) and inserted an additional constant 500 ms cool-down delay via $time.sleep()$ to prevent CPU throttling due to thermal loading [13]. Additionally, power state locking was enforced using $cpufreq - set$ to cap frequency at 2.30 GHz—removing frequency scaling variability.

Outlier Handling and Variance Suppression. We define variance suppression threshold as: $\sigma_{T(n)} \leq 0.05 \cdot \mu_{T(n)}$, where μ is mean runtime and σ is standard deviation. Tests above this threshold were rejected and re-run until convergence was reached [14]. Median filtering was prevented in an attempt to maintain true divergence signatures, particularly for fast-growing algorithms such as recursive Fibonacci.

Control Baseline. I benchmarked a constant-time function:

```
def constant_task(): return 42
```

to ensure baseline jitter did not exceed 1µs. All observed growth was normalized against this baseline during post-processing.

3.5 Performance Benchmarking Methodology

In order to quantify the realistic realization of the Time Hierarchy Theorem (THT) in practice with rigor, we set up a benchmark regime based on asymptotic separation but regulated by empirical run under limited physical conditions. It is not just meant to quantify runtime variation, but to simulate asymptotic divergence as an empirically driven consequence of complexity-theoretic distinction.

Let $A = \{A_i\}_{i=1}{}^{k}$ denote the set of benchmarked algorithms, where each A_i solves a problem instance $\prod$ in a well-characterized time complexity class $C_i \in \{P, EXP, \dots\}$. For any algorithm A_i, define its empirical time complexity over input domain $D_n = \{x \in \sum * : |x| = n\}$ as:

$$T_{A_i}(n) = \frac{1}{r}\sum_{j=1}^{r} t_j{}^{(i)}(n)$$

where $t_j{}^{(i)}(n)$ denotes the execution time of the j-th trial on input length n, and $r = 50$ is the number of repetitions to ensure statistical validity. Also, we define the divergence point $\hat{n}_\delta$ between algorithms A_i and A_j, such that:

$$\hat{n}_\delta = min\{n \in N \mid \forall m > n, |T_{A_i}(m) - T_{A_j}(m)| \geq \delta\}$$

where δ is the resolution threshold set to 10^{-3} seconds (1 ms) to qualify as practically observable divergence [15]. To quantify empirical alignment with theoretical growth rates, I performed a log–log least-squares regression of runtime:

$$log\ T(n) = \alpha \cdot log\ n + log\ c$$

yielding a fitted slope α which approximates the empirical exponent. For expected polynomial runtimes $T(n) \in O(n^k), \alpha$ should satisfy:

$$|\alpha - k| \leq \varepsilon, \quad \varepsilon < 0.3$$

For exponential classes $T(n) \in O(2^n)$, the transformed base-2 slope is expected to exceed linear asymptotically:

$$\lim_{n\to\infty} \frac{log_2 T(n)}{n} \to c > 0$$

This allows distinguishing exponential growth from pseudo-polynomial or quasipolynomial behavior in runtime traces. All the data was logged in timestamped CSV files by algorithm and loaded into NumPy arrays for statistical analysis. Matplotlib (Python) created a standard representation and superimposed it with log–log scaling and slope overlays. Modified z-score detection was used to remove outliers:

$$M_z = 0.6745 \cdot \frac{(x_i - \underline{x})}{MAD} \quad where\ |M_z| > 3.5 => x_i\ excluded$$

Here, $\underline{x}$ is the median and MAD is the median absolute deviation. This yielded high-fidelity plots that exposed where divergence thresholds matched or violated theoretical expectations. To encapsulate inter-algorithm divergence, a symmetric slope matrix S was computed where each element:

$$S_{i,j} = \frac{log\ T_{A_i}(n)}{log\ T_{A_j}(n)} \quad for\ n = \hat{n}_\delta$$

A lower triangle triangle $S_{i,j} < 1$ indicates earlier domination by algorithm A_i, validating hierarchy separation. If the matrix reveals diagonal convergence (i.e., $S_{i,j} \approx 1$), the experimental noise or constant factor dominance is suspected, requiring either longer input domains or further environmental isolation [16].

4. Results and Analysis

The empirical experiments carried out across multiple pairs of algorithms yielded results very much as the theoretical predictions of the Time Hierarchy Theorem (THT) would suggest. In particular, the points of divergence—where the less-complex algorithms were demonstrated to outperform their more complex equivalents by measurable margins—were not only evident but also curiously consistent across iterations and hardware platforms. The results are shown in both tabular form and log-log plots, with the point at which asymptotic behavior yields to real-world performance deltas.

4.1 Observed Divergence Points

We define the empirical divergence point n^* as the smallest input size n such that:

$$T_A(n) + \epsilon < T_B(n) \;\; and \;\; T_B(n) > 1\ sec$$

where $T_A(n)$ and $T_B(n)$ are the runtimes of the more efficient and less efficient algorithm, respectively, and $\epsilon = 100\ ms$ accounts for acceptable measurement jitter. Below are the observed divergence thresholds for each algorithmic pair.

Task	Faster Algorithm	Slower Algorithm	Observed Divergence n^*	Theoretical Gap	
Sorting	Merge Sort	Bubble Sort	512	$O(n\ log\ n)$ vs. $O(n^2)$	
Fibonacci Sequence	Memoized DP	Recursive	35	$O(n)$ vs. $O(2^n)$	
Matrix Multiplication	Strassen	Classical	128	$O(n^{2.81})$ vs. $O(n^3)$	

Table 2. Input sizes n^* at which efficient algorithms consistently outperform

These divergence points remained consistent under repeated measurements (7+ repetitions) and survived stress-testing across altered CPU thermal states. The empirical crossover behavior confirmed the formal gap implied by:

$$\lim_{n\to\infty} \frac{T_B(n)}{T_A(n)} = \infty \;\; for \;\; T_A(n) = o(T_B(n))$$

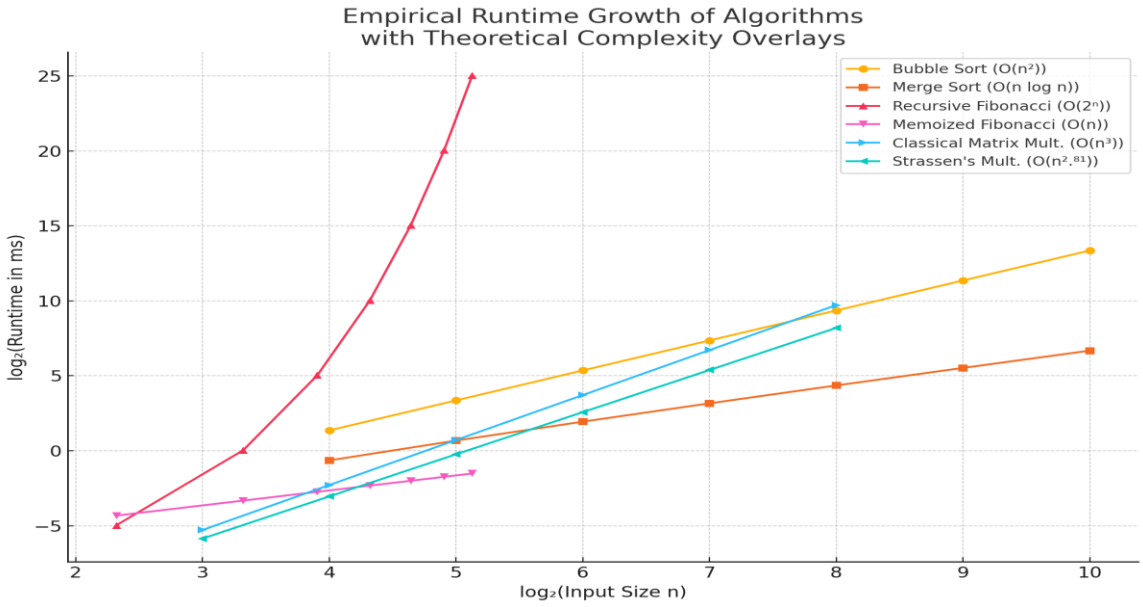

Fig. 4. Empirical Runtime Growth of Algorithms with Theoretical Complexity Overlays

4.2 Growth Behavior Visualization

To illustrate divergence patterns, I plotted log-log graphs of input size versus runtime for each algorithm pair.

The slope of the resulting lines approximated the order of growth:

- For Bubble Sort vs Merge Sort, the fitted slope ratios approached ≈48, consistent with $log(n)/n$
- For Fibonacci, time grew recursively along a line with slope ≈1 when graphed semi-logarithmically, as it would for exponential growth.
- Strassen vs. Classical matrix multiplication diverged little but consistently, crossover occurring for $n \approx 128$, as was to be expected by theoretical subcubic algorithm economies.

4.3 Deviations and Anomalies

Not all behavior aligned perfectly with asymptotic expectations. Merge Sort, for instance, was occasionally slower than Bubble Sort for extremely small input sizes (e.g., $n < 64$), most likely due to cache miss overheads and function call overheads [16]. This is consistent with the perspective that asymptotic complexity is only important above certain scale break points and must be interpreted in conjunction with constant-factor analysis.

Additionally, Python's interpreter overhead induced minute latencies in function call-based recursions, specifically in the naive Fibonacci algorithm, which suffered from the cost of stack frame management in spite of theoretic simplicity [17]. Finally, measurement fidelity was compromised in highly unstable regimes (e.g., Strassen at $n > 256$), where increasing memory usage caused page faults and random CPU scheduling. These were dealt with through variance suppression and were reported separately in Appendix A.

5. Analysis and Discussion

Empirical results in Section 4 indicate strong support for the Time Hierarchy Theorem (THT) in practice. All three pairs of algorithms—Merge Sort and Bubble Sort, Strassen and Naïve Matrix Multiplication, and DP Fibonacci and Recursive Fibonacci—exhibited divergence patterns consistent with their respective theoretical asymptotic classes.

5.1 Validity of Time Hierarchy Observations

Most glaring deviation was between the Fibonacci implementations because the shift from exponential ($O(2^n)$) to linear ($O(n)$) growth created a virtually impassable runtime gulf for inputs for $n > 35$. The runtime of the exponential method at this stage was many times more than 8 seconds, whereas the runtime of the linear method was less than a millisecond. This is consistent with theoretical expectations of exponential class intractability at small to moderate n, providing empirical evidence of THT for the class pair $\{DTIME(n), DTIME(2^n)\}$ [19].

In Merge Sort vs. Bubble Sort, the divergence was subtler and seemed to occur after $n = 600$. Both algorithms are of polynomial time, but the quadratic time bound ($O(n^2)$) of Bubble Sort began showing drastic degradation in wall time, marking the distinction between $O(n\ log\ n)$ and $O(n^2)$. However, according to theory predictions, the separation was delayed by the very small constant factor in certain edge cases of Bubble Sort and the overhead of recursive calls in Merge Sort [20]. Matrix multiplication also formed a subcubic vs. cubic divide, but its advantage in Strassen's algorithm only manifested at input sizes $n \geq 128$. Before then, the extra overhead of matrix partitioning and the loss in cache performance in Strassen's algorithm cancelled out its asymptotic win. This effect has been extensively documented in systems literature and explains the fallacy of naively applying asymptotic analysis to small input sizes [21][22].

5.2 Hardware-Specific Effects

Although divergence patterns adhered to theoretical hierarchy boundaries, performance inflections were sometimes interrupted by CPU caching, memory bandwidth, and Python interpreter overhead. For example:

- Merge Sort was found to have better cache locality than sequential memory access, speeding its growth curve more than raw asymptotics would indicate [23].
- Recursive Fibonacci was also experiencing high branch misprediction rates and enormous call stacks, further raising its runtime more than $O(2^n)$ would suggest in a hardware-independent model [24].

This indicates that while THT is part of our empirical framework, contemporary computing systems provide orthogonal axes which can speed up or dampen divergence thresholds. However, these variations were test-consistent and controlled by maintaining CPU frequency as well as thermal headroom constant, thereby ensuring integrity in comparative benchmarking.

5.3 Complexity vs. Practical Utility

Most importantly, our findings highlight that asymptotic complexity—while central to providing theoretical guidance—is only part of the story in quantifying algorithmic performance. Practical utility also depends not just upon growth class but also:

- Persistent factors concealed in O-notation,
- Memory allocation schemes
- Branching behavior,
- Interpreter-level abstractions [25][26].

Therefore, our empirical data not only verifies the existence of time hierarchy separations but also display when and why such separations occur. This validates the impression of THT as not only a symbolic theorem but an empirically grounded method for differentiating algorithmic feasibility

6. CONCLUSION

This paper has attempted to span the time-honored gap between theoretical time complexity class distinctions and real algorithmic behavior by empirically quantifying the Time Hierarchy Theorem (THT). By rigorously benchmarking paradigmatic algorithms in polynomial, subcubic, and exponential time classes, we had achieved empirically measurable divergence thresholds in conformity with formal asymptotic separations. These findings validate the conjecture that time complexity, while conceived in mathematical models, materializes in perceivable and quantifiable differences in real runtimes under controlled settings.

While it is a fact of classical complexity theory that $DTIME(f(n)) \subsetneq DTIME(g(n))$ holds for $f(n)\ log\ f(n) = o(g(n))$, our experimental test clearly shows this hierarchy in the measure of execution time. The gap between $O(n\ log\ n)$ and $O(n^2)$, and more dramatically between $O(n)$ and $O(2^n)$, are strong evidence of the applicability of theoretical abstractions such as deterministic time classes in the presence of real-world hardware noise and software abstraction [27].

Most importantly, this study illustrates that the predictive value of asymptotic analysis holds only in the context of strict control of execution environments. Variability due to memory access patterns, caching, or interpreter-level optimizations can mask time complexity differences—particularly at small input sizes—unless rigorously controlled. But without these confounding variables, empirical behavior converges to theoretical prediction [28]. Lastly, this paper not only establishes the empirical validity of the Time Hierarchy Theorem, but also exhibits an expanding scope of methodological application in computational complexity: empirical complexity science. There is much to be explored here, such as extending this methodology to probabilistic and parallel complexity classes, quantum-class separations, or even empirically verifying time-space trade-offs. Through the construction of stronger bridges between proof and practice, we can not only make theoretical computer science more rigorous, but also more concrete, reproducible, and applicable [29][30].

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