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A case study on Lyapunov stability analysis of nonlinear systems with ordinary differential equations

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Abstract- ***The main concern of this paper is the stability analysis of nonlinear systems. For evaluating the stability of a system around an equilibrium point, firstly need to understand the basic concepts of stability theory and then explore different methods to apply on the systems to check and verify the required conditions for it. This work emphasizes on the conditions needed to guarantee asymptotic stability in nonlinear dynamic autonomous systems of continuous-time.***

Index Terms- *Nonlinear ODE, Energy, Lyapunov Function, Asymptotic Stability.*

1. INTRODUCTION

Stability theory plays a central role in systems theory and engineering. It is essential to have a stable system for any given system as an unstable control system is useless. Lyapunov stability theory provides an effective method for studying the stability of nonlinear systems. Basically, two approaches are included in the concepts of Lyapunov stability. The first one is the Lyapunov indirect method and the second method is the Lyapunov direct method.

A Russian mathematician Aleksandra M. Lyapunov [1] originated Lyapunov functions in 1892 for studying the problem of stability. The significance of this method is based on the well-known fact that the total energy in a system is either constant or decreasing towards the equilibrium position. Idea behind the Lyapunov indirect method is linearizing the system around a given point and achieving local stability with small stability regions. On the other hand, the Lyapunov direct method [2] can be applied directly to a nonlinear system without any linearization and achieves global stability.

The paper is organized as follows. Section 2 provides a preliminary idea about Lyapunov stability theory. Section 3 presents the indirect method theorems with its mathematical modeling for the 2nd order and 3rd order nonlinear systems.

Section 4 shows direct method theorem and its application on 2nd order and 3rd order nonlinear systems. Sections 5 and 6 are reserved for analysis and comparison of both methods and then finally the conclusion is provided in Section 7.

2. LYAPUNOV STABILITY THEORY

2.1 Definition of Stability in sense of Lyapunov

Consider the system

$$\dot{x}(t) = Ax(t) \tag{2.1}$$

Lyapunov stability is defined about the equilibrium state which is located at the origin and x(0). Let S(R) be spherical region of radius R>0 around the origin, where S(R) consists of point x satisfying ||x||<R.

The origin is said to be stable in the sense of Lyapunov or simple stable if corresponding to each S(R) there is an S(r) such that the solutions starting in s(r) do not leave S(R).

- Stable at origin if for every initial state $x(t_o)$ which is near to origin, $x(t)$ remains near the origin for all ‘t’.
- Asymptotically stable [6], if $x(t)$ approaches to the origin as ‘t’ tending to infinity.
- Asymptotically stable at large i.e. system for every initial states, if system approaches to origin as time approaches to infinity.

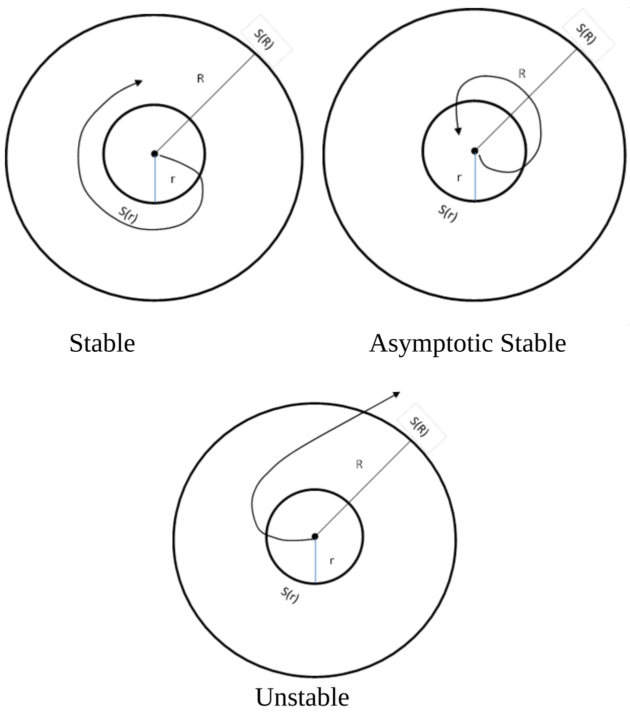

Figure 1: Lyapunov Stability Theory

According to Lyapunov, stability depends upon whether the function is positive definite or negative definite. Suppose V(x) is a function that basically describes the energy of the system. The energy of the system can be zero or it should be greater than zero but it cannot be negative.

2.2 Basic Concepts for Positive and Negative Definiteness

$V(x_1, x_2, x_3, \ldots\ldots\ldots, x_n) > 0$ For any nonzero value of $x_1, x_2, x_3, \ldots., x_n$. Positive definite

$V(x_1, x_2, x_3, \ldots\ldots\ldots, x_n) < 0$ For any nonzero value of $x_1, x_2, x_3, \ldots., x_n$. Negative definite

$V(x_1, x_2, x_3, \ldots\ldots\ldots, x_n) \geq 0$ For any nonzero value of $x_1, x_2, x_3, \ldots., x_n$. Positive semi-definite

$V(x_1, x_2, x_3, \ldots\ldots\ldots, x_n) \leq 0$ For any nonzero value of $x_1, x_2, x_3, \ldots., x_n$. Negative semi definite

$V(x_1, x_2, x_3, \ldots\ldots\ldots, x_n)$ if we can’t say anything about the sign of the function for any nonzero value of $x_1, x_2, x_3, \ldots., x_n$. Indefinite.

Examples

- $V(x_1, x_2) = x_1^2 + x_2^2$ (2.2)

Putting any non-zero values (0, 0), (0, 3), (1, 0), (2, 1) which will give positive value. So we can say that it is +ve definite.

- $V(x_1, x_2) = -x_1^2 - (3x_1 + x_2)^2$ (2.3)

Taking different non-zero values (0, 1), (2, 2), $(x_1, -3x_1)$ which will give negative value. So, we can say that it is –ve definite.

- $V(x_1, x_2) = (x_1 + x_2)^2$ (2.4)

For all values it will be positive except $x_1 = -x_2$ in that case $V(x_1, x_2) = 0$, so it will satisfy rule3. So, we can say that it is +ve semi-definite.

- $V(x_1, x_2) = x_1^2 + 2x_1x_2$ (2.5)

We cannot define the sign of the function so it will be indefinite.

2.3 Global and Local Stability

Suppose a system is asymptotically stable for any initial condition of the system at $t \to \infty$, if it falls on the equilibrium state i.e. $\partial_a \to 0$, in that case the system is known as globally asymptotically stable.

If not fall on equilibrium state then known as local stable.

3. LYAPUNOV’S INDIRECT METHOD

3.1 General Idea

This section is focused on the problem of exploring stability properties of an equilibrium state of a nonlinear system based on its linearization about the given equilibrium. A method is developed which allows us to determine the stability of the nonlinear system about the equilibrium point on the basis of the linearized system. The method is known as Lyapunov's indirect method or Lyapunov's first method. In this, the equilibrium point to be tested for stability is the origin for simplicity of notation.

3.2 Basic Theorems of Stability

Considering a nonlinear system for stability analysis consequently linearize it about the equilibrium point and check the system's eigenvalues to examine stability. The following theorem provides the necessary conditions for examining the stability of a linearized system. So we can draw conclusions about the local stability of an equilibrium point of a nonlinear system.

- If the eigenvalues of the A matrix in the linearized system have negative real parts, the nonlinear system is asymptotically stable about the equilibrium point.

- If at least one eigenvalue of the A matrix in the linearized system has positive real part, the nonlinear system is unstable about the equilibrium point.
- If at least one eigenvalue of the A matrix in the linearized system has zero real part, the test is inconclusive. The linear approximation is insufficient to determine stability. However, methods exist to include higher-order terms (center manifold technique).

3.3 Construction of Quadratic Lyapunov Functions

It is observed that all the functions have been taken in section 2.2 basically are the functions of two variables. These variables are basically state variables in a system. Now, if we only consider 2^{nd} order system then this method is effective but this won't work for more than 2 variables. In that case the system will give by

$$\dot{x} = Ax \quad (3.1)$$

Let us consider the energy function

$$V(x) = x'Px \quad (3.2)$$

This is quadratic form. In this, P has to be identical/ symmetrical matrix i.e. $P_{nm} \equiv P_{mn}$

3.4 Lyapunov function for Liner system

The liner system given in system is globally asymptotically stable at the origin, if and only if, for any given symmetric +ve definite matrix Q, there exist a symmetric +ve definite matrix P that satisfy the matrix equation.

Then

$$\dot{V}(x) = \dot{x}'Px + x'P\dot{x} \quad (3.3)$$

$$\dot{V}(x) = (Ax)'Px + x'P(Ax) \quad (3.4)$$

$$\dot{V}(x) = x'A'Px + x'PAx$$

$$\dot{V}(x) = x'(A'P + PA)x$$

$$\dot{V}(x) = x'Q\,x \quad (3.5)$$

Where,

$$Q = (A'P + PA) \quad (3.6)$$

Now, according to definition of stability $\dot{v}(x)$ should be –ve definite. So, one can check Q will be –ve definite or not by equating this with standard negative definite matrix which is –I. We can also take –ve semi definite also in case of marginal stability.

$$Q = (A'P + PA) = -I \quad (3.7)$$

Here, A= $n \times n$ real constant matrix
I = Identity matrix
P= Symmetric Matrix

So, it can be find out P from here and after that we need to check

$$V(x) = x'Px$$

For which we have to check all minors of P if V(x) –ve definite then the system will be unstable if +ve then stable.

Lyapunov Function for 2^{nd} Order Equation

$$V(x) = x'Px$$

$$V(x) = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$V(x) = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} P_{11}x_1 + P_{12}x_2 \\ P_{12}x_1 + P_{22}x_2 \end{bmatrix}$$

$$V(x) = P_{11}{x_1}^2 + P_{12}x_1x_2 + P_{12}x_1x_2 + P_{22}{x_2}^2$$

$$V(x) = P_{11}{x_1}^2 + P_{22}{x_2}^2 + 2P_{12}x_1x_2 \quad (3.8)$$

Lyapunov Function for 3^{rd} Order Equation

$$V(x) = x'Px$$

$$V(x) = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{12} & P_{22} & P_{23} \\ P_{13} & P_{23} & P_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$V(x) = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} P_{11}x_1 + P_{12}x_2 + P_{13}x_3 \\ P_{12}x_1 + P_{22}x_2 + P_{23}x_3 \\ P_{13}x_1 + P_{23}x_2 + P_{33}x_3 \end{bmatrix}$$

$$V(x) = P_{11}{x_1}^2 + P_{12}x_1x_2 + P_{13}x_1x_3 + P_{12}x_1x_2 + P_{22}{x_2}^2 + P_{23}x_2x_3 + P_{13}x_1x_3 + P_{23}x_2x_3 + P_{33}{x_3}^2$$

$$V(x) = P_{11}{x_1}^2 + P_{22}{x_2}^2 + P_{33}{x_3}^2 + 2P_{12}x_1x_2 + 2P_{13}x_1x_3 + 2P_{23}x_2x_3 \quad (3.9)$$

3.5 Implementation of Indirect Theorem on Linear System Equations

3.5.1 Checking Stability of 2^{nd} Order System

Test System1

$$\ddot{x} + \dot{x} + x = 0 \quad (3.10)$$

$$x = x_1$$

$$\dot{x} = x_2$$

$$\dot{x}_1 = x_2 \quad (3.11)$$

$$\dot{x}_2 = -x_1 - x_2 \quad (3.12)$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Let

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

In quadratic format it should be

$$P_{21} \equiv P_{12} \quad (3.13)$$

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} \quad (3.14)$$

Now, according to definition of stability $\dot{v}(x)$ should be –ve definite

$$\dot{V}(x) = x' Q\, x = -I$$

$$\dot{V}(x) = x'(A'P + PA)x = -I \quad (3.15)$$

$$\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} + \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -P_{12} & -P_{22} \\ (P_{11} - P_{12}) & (P_{12} - P_{22}) \end{bmatrix} + \begin{bmatrix} -P_{12} & (P_{11} - P_{12}) \\ -P_{22} & (P_{12} - P_{22}) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$-2P_{12} = -1 \quad (3.16)$$

$$-P_{22} + P_{11} - P_{12} = 0 \quad (3.17)$$

$$2(P_{12} - P_{22}) = -1 \quad (3.18)$$

Solving equations (3.16), (3.17) and (3.18)

$$P_{11} = 3/2$$

$$P_{12} = 1/2$$

$$P_{22} = 1$$

$$P = \begin{bmatrix} 3/2 & 1/2 \\ 1/2 & 1 \end{bmatrix} \quad (3.19)$$

Now, checking all minors

$$P_1 = \frac{3}{2} > 0$$

$$P_2 = \frac{3}{2} - \frac{1}{4} = \frac{5}{4} > 0$$

Therefore, the system is stable.

MATLAB Simulation

```
A = [0 1;-1 -1];
P = [3/2 1/2; 1/2 1];
X = lyap (A, P)
```

Lyapunov Solution is Matrix X =

```
   2.5000   -0.7500
  -0.7500    1.2500
```

Eigen values of Lyapunov:

```
   0.8987
   2.8513
```

The System is Positive Definite and hence stable.

3.5.2 Checking Stability of 3rd Order System

Test System 2

$$f(x) = 5x_1^2 + 2x_2^2 + x_3^2 + 2x_1x_2 - 2x_2x_3 - 4x_3x_1 \quad (3.20)$$

It is 3rd order system. Now, it needs to express in quadratic form according to equation (3.9).

$$V(x) = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}\begin{bmatrix} 5 & 1 & -2 \\ 1 & 2 & -1 \\ -2 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad (3.21)$$

Now we check,

- If all the minors are greater than zero then the system is positive definite.
- If all the minors are less than zero then the system is negative definite.
- If some of the minors is equals to zero and rest of minors are greater than zero then it is +ve semi-definite.
- If some of the minors is equals to zero and rest of minors are less than zero then it is -ve semi-definite.
- If some of the minors are positive and some are negative then it is indefinite.

$$P = \begin{bmatrix} 5 & 1 & -2 \\ 1 & 2 & -1 \\ -2 & -1 & 1 \end{bmatrix}$$

Using Sylvester's test

$$P_1 = 5 > 0$$

$$P_2 = (10 - 1) = 9 > 0$$

$$P_3 = 5(2 - 1) - 1(0 - 2) - 2(-1 + 4) = 1 > 0$$

Successive principal minor of the matrix P are +ve. So, V(x) is positive definite. Hence, the system is stable.

3.5.3 Finding Value of Unknown Variable When System is Stable

Test System 3

$$\dot{x}_1 = x_2 \quad (3.22)$$

$$\dot{x}_2 = -2x_2 + x_3 \quad (3.23)$$

$$\dot{x}_3 = -Kx_1 - x_3 + Ku(t) \quad (3.24)$$

Now, it needs to express in quadratic form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -2 & 1 \\ -K & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ K \end{bmatrix}$$

Here, matrix A

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -2 & 1 \\ -K & 0 & -1 \end{bmatrix}$$

From equation (3.7),

$$(A'P + PA) = -I \tag{3.25a}$$

$$\begin{bmatrix} 0 & 1 & -K \\ 0 & -2 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} P_{11} & P_1 & P_2 \\ P_1 & P_{22} & P_3 \\ P_2 & P_3 & P_{33} \end{bmatrix} + \begin{bmatrix} P_{11} & P_1 & P_2 \\ P_1 & P_{22} & P_3 \\ P_2 & P_3 & P_{33} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & -2 & 1 \\ -K & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -KP_2 & -KP_3 & -KP_{33} \\ (P_{11} - 2P_1) & (P_1 - 2P_{22}) & (P_2 - 2P_3) \\ (P_1 - P_2) & (P_{22} - P_3) & (P_3 - P_{33}) \end{bmatrix} + \begin{bmatrix} -KP_2 & (P_{11} - 2P_1) & (P_1 - P_2) \\ -KP_3 & (P_1 - 2P_{22}) & (P_{22} - P_3) \\ -KP_{33} & (P_2 - 2P_3) & (P_3 - P_{33}) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -2KP_2 & (P_{11} - 2P_1 - KP_3) & (P_1 - P_2 - KP_{33}) \\ (P_{11} - 2P_1 - KP_3) & (P_1 - 2P_{22}) \times 2 & (P_2 + P_{22} - 3P_3) \\ (P_1 - P_2 - KP_{33}) & (P_2 + P_{22} - 3P_3) & (P_3 - P_{33}) \times 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{3.25b}$$

Equating both matrices,

$$2KP_2 = 0 \tag{3.26}$$

$$P_{11} - 2P_1 - KP_3 = 0 \tag{3.27}$$

$$P_1 - P_2 - KP_{33} = 0 \tag{3.28}$$

$$P_1 - 2P_{22} = 0 \tag{3.29}$$

$$P_2 + P_{22} - 3P_3 = 0 \tag{3.30}$$

$$P_3 - P_{33} = -\frac{1}{2} \tag{3.31}$$

By solving these equations,

$$P_{33} = \frac{3}{(6-K)} \tag{3.32}$$

$$P_{22} = \frac{3K}{2(6-K)} \tag{3.33}$$

$$P_3 = \frac{K}{2(6-K)} \tag{3.34}$$

$$P_1 = \frac{6K}{2(6-K)} \tag{3.35}$$

$$P_{11} = \frac{K^2+12}{2(6-K)} \tag{3.36}$$

$$P = \begin{bmatrix} \frac{K^2+12}{2(6-K)} & \frac{6K}{2(6-K)} & 0 \\ \frac{6K}{2(6-K)} & \frac{3K}{2(6-K)} & \frac{K}{2(6-K)} \\ 0 & \frac{K}{2(6-K)} & \frac{3}{(6-K)} \end{bmatrix} \tag{3.37}$$

For each minor should be greater than 0 so it need to be $K > 0$ then $(6 - K) > 0$ also to be >0.

$$(6 - K) > 0$$

$$K < 6$$

$$0 < K < 6 \tag{3.38}$$

This is the range of K for stability.

Let K=1 for MATLAB Verification.

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -2 & 1 \\ -1 & 0 & -1 \end{bmatrix}$$

$$P = \begin{bmatrix} \frac{6}{5} & \frac{3}{5} & 0 \\ \frac{3}{5} & \frac{3}{10} & \frac{1}{10} \\ 0 & \frac{1}{10} & \frac{3}{5} \end{bmatrix}$$

MATLAB Simulation

```
A = [0 1 0;-0 -2 1;-1 0 -1];
P = [6/5 3/5 0;3/5 3/10 1/10;0 1/10 3/5];
X = lyap (A, P)
```

Lyapunov Solution is Matrix X =

```
 3.6000  -0.6000  -2.4500
-0.6000   0.6500   1.1500
-2.4500   1.1500   2.7500
```

Eigen values of Lyapunov:

```
0.0265
1.0373
5.9362
```

The System is Positive Definite and hence stable

3.6 Construction of Lyapunov Function for nonlinear systems using Krasovskii Method

Consider the nonlinear autonomous n-order system. This system might be described by one nonlinear *n*-order equation or by a set on n first-order nonlinear differential equations (1) or matrix equation (2).

$$\dot{x}_1 = f_1(x_1, x_2, x_3, \ldots\ldots, x_n)$$
$$\dot{x}_2 = f_2(x_1, x_2, x_3, \ldots\ldots, x_n)$$
$$\ldots\ldots\ldots\ldots$$
$$\dot{x}_n = f_n(x_1, x_2, x_3, \ldots\ldots, x_n)$$

$$\dot{x} = f(x) \quad (3.39)$$

$$V(x) = f^T(x)Pf(x) \quad (3.40)$$

$$\dot{V}(x) = \dot{f}^T Pf(x) + f^T P\dot{f}(x) \quad (3.41)$$

$$\dot{f}(x) = \frac{\partial f(x)}{\partial x} \cdot \frac{\partial x}{\partial t} = J(x)f(x) \quad (3.42)$$

From (3.42)

$$\dot{V}(x) = J^T(x)f^T(x)Pf(x) + f^T(x)PJ(x)f(x)$$

$$\dot{V}(x) = f^T(x)[J^T(x)P + PJ(x)]f(x) \quad (3.43)$$

Compare with

$$\dot{V}(x) = -x^T Q\, x \quad (3.44)$$

$$Q = -[J^T(x)P + PJ(x)] \quad (3.45)$$

Where,

$$J(x) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix} \quad (3.46)$$

Since, V(x) is +ve definite, for the system asymptotically stable at the origin. $\dot{V}(x)$ Should be –ve definite or Q should be + ve definite.

3.7 Implementation of Indirect theorem on Nonlinear System Equations

3.7.1 Finding value of unknown variable when system is stable

Test System 4

$$\dot{x}_1 = f_1(x) = x_2 \quad (3.47)$$
$$\dot{x}_2 = f_2(x) = -x_2 - Kx_1^3 \quad (3.48)$$

Now, Calculate J(X)

$$J(X) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} \quad (3.49a)$$

From System Equations

$$\frac{\partial f_1}{\partial x_1} = 0$$

$$\frac{\partial f_1}{\partial x_2} = 1$$

$$\frac{\partial f_1}{\partial x_1} = -3kx_1^2$$

$$\frac{\partial f_2}{\partial x_2} = -1$$

$$J(X) = \begin{bmatrix} 0 & 1 \\ -3kx_1^2 & -1 \end{bmatrix}$$

(3.49b)

Let

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

If P is +ve definite then

$$|P_{11}| > 0$$

$$\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} > 0$$

i.e.

$$P_{11}P_{22} - P_{21}^2 > 0 \quad (3.50)$$

Now calculate

$$Q = -[J^T(X)P + PJ(X)] \quad (3.51)$$

$$Q = -\left\{\begin{bmatrix} 0 & -3kx_1^2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} + \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -3kx_1^2 & -1 \end{bmatrix}\right\}$$

$$Q = \left\{\begin{bmatrix} -3kx_1^2P_{12} & -3kx_1^2P_{22} \\ P_{11} - P_{12} & P_{12} - P_{22} \end{bmatrix} + \begin{bmatrix} -3kx_1^2P_{12} & P_{11} - P_{12} \\ -3kx_1^2P_{22} & P_{12} - P_{22} \end{bmatrix}\right\}$$

$$Q = -\left\{\begin{bmatrix} -6kx_1^2P_{12} & P_{11} - P_{12} - 3kx_1^2P_{22} \\ P_{11} - P_{12} - 3kx_1^2P_{22} & 2(P_{12} - P_{22}) \end{bmatrix}\right\}$$

$$Q = \begin{bmatrix} 6kx_1^2P_{12} & -P_{11} + P_{12} + 3kx_1^2P_{22} \\ -P_{11} + P_{12} + 3kx_1^2P_{22} & -2(P_{12} - P_{22}) \end{bmatrix} \quad (3.52)$$

For the system asymptotically stable, Q should be +ve definite.

Thus,

$$6kx_1^2P_{12} > 0$$

And

$$Q = \begin{bmatrix} 6kx_1^2P_{12} & -P_{11} + P_{12} + 3kx_1^2P_{22} \\ -P_{11} + P_{12} + 3kx_1^2P_{22} & -2(P_{12} - P_{22}) \end{bmatrix} > 0$$

$$-12kx_1^2P_{12}(P_{12} - P_{22}) - (P_{11} - P_{12} - 3kx_1^2{P_{22}}^2) > 0$$

$$12kx_1^2P_{12}(P_{22} - P_{12}) > (P_{11} - P_{12} - 3kx_1^2{P_{22}}^2)$$

(3.53)

If

$$P_{12} > 0$$

Then

$$x_1^2 > 0$$

Choose

$$P_{11} = P_{12}$$

And

$$P_{22} = \beta P_{12}$$

Put these values in equation (3.53)

$$12kx_1^2P_{12}(\beta P_{12} - P_{12}) > (P_{12} - P_{12} - 3kx_1^2\beta P_{12})^2$$

$$P_{12}{}^2[12kx_1^2(\beta-1)] > 9(kx_1^2)^2\beta^2 P_{12}{}^2$$

$$12(\beta-1) > 9kx_1^2\beta^2$$

$$x_1^2 < \frac{12(\beta-1)}{9k\beta^2}$$

$$x_1^2 < \frac{4}{3k}\left(\frac{1}{\beta}-\frac{1}{\beta^2}\right)$$

So, it can be considered that tangent value of x_1 occurs when $\beta = 2$, therefore

$$x_1^2 < \frac{1}{3k}$$

$$-\frac{1}{\sqrt{3k}} < x_1 < \frac{1}{\sqrt{3k}} \qquad (3.54)$$

This is the range of x_1 for stability.

3.7.2 Checking Stability of 3rd Order Nonlinear System

Test System 5

$$\dot{x}_1 = x_2 \qquad (3.55)$$

$$\dot{x}_2 = -6x_1 - 5x_2 \qquad (3.56)$$

$$\dot{x}_3 = -2x_2 + x_3 \qquad (3.57)$$

Now Calculate J(x)

$$J(x) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} \end{bmatrix} \qquad (3.58)$$

$$\frac{\partial f_1}{\partial x_1} = 0, \quad \frac{\partial f_1}{\partial x_2} = 1, \quad \frac{\partial f_1}{\partial x_3} = 0$$

$$\frac{\partial f_2}{\partial x_1} = -6, \quad \frac{\partial f_2}{\partial x_2} = -5, \quad \frac{\partial f_2}{\partial x_3} = 0$$

$$\frac{\partial f_3}{\partial x_1} = -2, \quad \frac{\partial f_3}{\partial x_2} = 0, \quad \frac{\partial f_3}{\partial x_3} = 1$$

$$J(x) = \begin{bmatrix} 0 & 1 & 0 \\ -6 & -5 & 0 \\ -2 & 0 & 1 \end{bmatrix} \qquad (3.59)$$

All Minors

$$J_{11} = \begin{bmatrix} -5 & 0 \\ 0 & 1 \end{bmatrix} = -5, \qquad J_{12} = \begin{bmatrix} -6 & 0 \\ -2 & 1 \end{bmatrix} = -6$$

$$J_{13} = \begin{bmatrix} -6 & -5 \\ -2 & 0 \end{bmatrix} = -10$$

$$J_{21} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1, \qquad J_{22} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} = 0$$

$$J_{23} = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} = 2$$

$$J_{31} = \begin{bmatrix} 1 & 0 \\ -5 & 0 \end{bmatrix} = 0, \qquad J_{32} = \begin{bmatrix} 0 & 0 \\ -6 & 0 \end{bmatrix} = 0$$

$$J_{33} = \begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix} = 6$$

$$M = \begin{bmatrix} -5 & -6 & 10 \\ 1 & 0 & 2 \\ 0 & 0 & 6 \end{bmatrix}$$

Because some of the minors are positive and some are negative then system is indefinite and it is not possible to decide about stability for this Lyapunov function.

4. Lyapunov direct method

4.1 General Idea

Lyapunov's direct method is the second method of Lyapunov. It is one of the effective and powerful classical methods used for reviewing asymptotic performance and stability of the dynamical systems using ordinary differential equations.

The vital concept of energy dissipation is taken as base for Lyapunov's direct method which says any system will eventually stretch to an equilibrium point and stay at that point if and only if the total energy of a system is continuously dissipating.

This method involves two steps. The first step is choosing a suitable scalar function which is referred as the Lyapunov function [3, 4]. Then evaluate its first-order time derivative along the trajectory of the system in the second step. If the derivative of a Lyapunov function is decreasing along the system trajectory as time increases that means the system energy is dissipating and the system will settle down and become stable [5].

This approach is very useful since it allows evaluating the stability of equilibrium points of a system without solving the system differential equations.

4.2 Basic Theorem's of Stability

Consider the nonlinear autonomous n-order system. This system might be described by one nonlinear *n*-order equation or by a set on n first-order nonlinear differential equations (1) or matrix equation (2).

$$\dot{x}_1 = f_1(x_1, x_2, x_3, \ldots\ldots\ldots, x_n)$$
$$\dot{x}_2 = f_2(x_1, x_2, x_3, \ldots\ldots\ldots, x_n)$$
$$\ldots\ldots\ldots\ldots\ldots..$$
$$\dot{x}_n = f_n(x_1, x_2, x_3, \ldots\ldots\ldots, x_n)$$

Or

$$\dot{x} = f(x) \qquad (4.1)$$

The vector **x** is the state vector, and its elements are state variables. The origin **x** = **0** of the state space will be assumed to be an equilibrium solution.

The Lyapunov function $V(x_1, x_2, x_3, \ldots\ldots, x_n)$ is a scalar function of the state variables and its positive definite. Now let

$$\dot{V}(x,t) = \frac{\partial v}{\partial t} \quad (4.2)$$

$$\dot{V}(x,t) = \frac{\partial v}{\partial x_1}\frac{\partial x_1}{\partial t} + \frac{\partial v}{\partial x_2}\frac{\partial x_2}{\partial t} + \cdots + \frac{\partial v}{\partial x_n}\frac{\partial x_n}{\partial t} \quad (4.3)$$

be calculated by substituting (1). If $\dot{V}(x,t)$ always negative, then apparently v decreases continuously, and the state must end up in the origin of the state space, implying asymptotic stability. To develop these concepts, the following definitions re used for v(x,t) and $\dot{v}$ (x,t)

- If V(x, t) is positive definite, and $\dot{V}(x,t)$is negative definite, then the system is globally asymptotically stable.
- If V(x, t) is positive definite, and $\dot{V}(x,t)$is negative semi definite, then the system is globally stable.
- If $\dot{V}(x,t)$is positive definite or semi definite, then the system is unstable.
- $\dot{V}(x,t)$is indefinite then it is not possible to decide about stability.

4.3 Implementation of Direct Theorem on Linear Systems

4.3.1 Checking Stability of 2nd Order System

Test System 6

$$\ddot{e} + (k+\alpha^2)e = 0 \quad (4.4)$$

Let,

$$x_1 = e$$
$$x_2 = \dot{e}$$

So,

$$\dot{x}_1 = \dot{e} = x_2 \quad (4.5)$$
$$\dot{x}_2 = \ddot{e} = -(k+\alpha^2)e$$
$$\dot{x}_2 = -(k+\alpha^2)x_1 \quad (4.6)$$

The main purpose is to choose a function which is positive definite.

$$V(X) = x_1^2 + x_2^2 \quad (4.7)$$

V(X) is always greater than 0

$$V(X) > 0 \ \text{ if } x_1, x_2 > 0$$

So, it is a simple positive definite function.

$$\frac{\partial V}{\partial x_1} = 2x_1 \quad (4.8)$$

$$\frac{\partial V}{\partial x_2} = 2x_2 \quad (4.9)$$

Checking Stability

$$\dot{V}(X) = \frac{\partial V}{\partial x_1}\cdot\frac{\partial x_1}{\partial t} + \frac{\partial V}{\partial x_2}\frac{\partial x_2}{\partial t} \quad (4.10)$$

Now, substituting the values in equation,

$$\dot{V}(X) = 2x_1(x_2) + 2x_2\{-(k+\alpha^2)x_1\}$$

$$\dot{V}(X) = 2x_1(x_2) - 2x_1x_2(k+\alpha^2) \quad (4.11)$$

Since, this is a function of +ve and –ve values of it, hence it is indefinite function.

Means $V(X) = x_1^2 + x_2^2$ is not suitable Lyapunov function for this system.

Now, we choose

$$V(X) = P_1x_1^2 + P_2x_2^2 \quad (4.12)$$

Where, $P_1 > 0$ and $P_2 > 0$

$$\frac{\partial V}{\partial x_1} = 2P_1x_1 \quad (4.13)$$

$$\frac{\partial V}{\partial x_2} = 2P_2x_2 \quad (4.14)$$

$$\dot{V}(X) = \frac{\partial V}{\partial x_1}\cdot\frac{\partial x_1}{\partial t} + \frac{\partial V}{\partial x_2}\frac{\partial x_2}{\partial t} \quad (4.15)$$

Now, substituting the values in equation,

$$\dot{V}(X) = 2P_1x_1x_2 + 2P_2x_2\{-(k+\alpha^2)x_1\}$$

$$\dot{V}(X) = 2P_1x_1x_2 - 2P_2x_1x_2(k+\alpha^2) \quad (4.16)$$

If,

$$P_1 = P_2(k+\alpha^2)$$

Then,

$$\dot{V}(X) = 0$$

So, it can be considered the equilibrium state of the system is stable in the sense of Lyapunov.

4.4 Implementation of theorem on Nonlinear Systems

4.4.1 Checking Stability of 2nd Order System

Test system 7

$$\dot{x}_1 = x_2 \quad (4.17)$$
$$\dot{x}_2 = -2x_1 - x_2^3 \quad (4.18)$$

The main purpose is to choose a function which is positive definite.

$$V(X) = x_1^2 + x_2^2 \quad (4.19)$$

V(X) is always greater than 0

$$V(X) > 0 \ \text{ if } x_1, x_2 > 0$$

So, it is a simple positive definite function.

Checking Stability

$$\dot{V}(X) = \frac{\partial V}{\partial x_1}\cdot\frac{\partial x_1}{\partial t} + \frac{\partial V}{\partial x_2}\frac{\partial x_2}{\partial t} \quad (4.20)$$

From (4.19)

$$\frac{\partial V}{\partial x_1} = 2x_1$$

$$\frac{\partial V}{\partial x_2} = 2x_2$$

Now, substituting the values in equation (4.20)

$$\dot{V}(X) = 2x_1(\dot{X}_1) + 2x_2(\dot{X}_2)$$

$$\dot{V}(X) = 2x_1(x_2) + 2x_2(-2x_1 - x_2^3)$$

$$\dot{V}(X) = 2x_1x_2 - 4x_1x_2 - 2x_2^4$$

$$\dot{V}(X) = -2x_2 - 2x_2^4 \quad (4.21)$$

So, it can be verified that if $x_1, x_2 > 0$ then $\dot{V}(X) < 0$

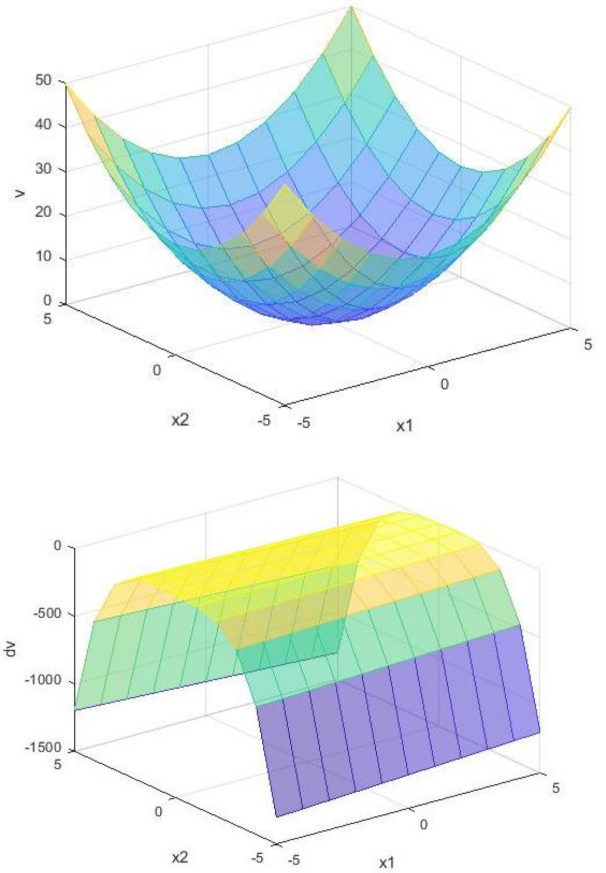

Figure 2: 3D Graphics of lyapunov function and Derivative of Lyapunov Function for nonlinear System

Therefore, it can be considered that the derivative of lyapunov function is –ve definite which shows that V(x) is continuously decreasing along any trajectory with respect of time. Hence the system is stable in the sense of Lyapunov. The 3D Graphics for this nonlinear System is shown in Figure 2 of lyapunov function is positive while Derivative of Lyapunov Function is continuously decreasing as time increases which confirms that tests system is stable.

4.4.2 Checking Stability of 3rd Order Nonlinear System

Test system 8

$$\dot{x}_1 = x_2 \tag{4.22}$$

$$\dot{x}_2 = -6x_1 - 5x_2 \tag{4.23}$$

$$\dot{x}_3 = -2x_2 + x_3 \tag{4.24}$$

Let choose a positive definite Lyapunov function.

$$V(X) = \alpha_1 x_1^2 + \alpha_2 x_2^2 + \alpha_3 x_3^2 \tag{4.25}$$

Here α_1, α_2 and α_3 are positive constants.

$$V(X) > 0 \text{ if } x_1, x_2, x_3 > 0$$

$$\frac{\partial V}{\partial x_1} = 2\alpha_1 x_1$$

$$\frac{\partial V}{\partial x_2} = 2\alpha_2 x_2$$

$$\frac{\partial V}{\partial x_3} = 2\alpha_3 x_3$$

$$\dot{V}(X) = \frac{\partial V}{\partial x_1} \cdot \frac{\partial x_1}{\partial t} + \frac{\partial V}{\partial x_2}\frac{\partial x_2}{\partial t} + \frac{\partial V}{\partial x_3} \cdot \frac{\partial x_3}{\partial t} \tag{4.26}$$

$$\dot{V}(X) = 2\alpha_1 x_1 (x_2) + 2\alpha_2 x_2 (-6x_1 - 5x_2) + 2\alpha_3 x_3 (-2x_2 + x_3)$$

$$\dot{V}(X) = (2\alpha_1 - 12\alpha_2) x_1 x_2 - 10\alpha_2 x_2{}^2 - (4\alpha_3 x_2 - 2\alpha_3 x_3) x_3 \tag{4.27}$$

If we choose $\alpha_1 = 6, \alpha_2 = 1$ and $\alpha_3 = 3$ then the $\dot{V}(X)$ becomes

$$\dot{V}(X) = -10x_2{}^2 + 6x_3{}^2 - 12x_2 x_3 \tag{4.28}$$

So, it can be verified that if $x_1, x_2, x_3 > 0$ then $\dot{V}(X) < 0$

If choose $x_2 = 2, x_3 = 3$

$$\dot{V}(X) = -40 + 54 - 72 = -58$$

If choose $x_2 = 2, x_3 = 7$

$$\dot{V}(X) = -40 + 294 - 168 = 86$$

Therefore, it can be considered that the derivative of lyapunov function is indefinite and it is not possible to decide about stability for this Lyapunov function.

5. Performance Analysis

In this work, stability of different linear and nonlinear systems is tested on the basis of Lyapunov theorems. Indirect and direct methods are implemented on various test systems to review asymptotic performance and stability of the dynamical systems using ordinary differential equations. Stability depends upon whether the Lyapunov function is positive definite or negative definite which basically describes the energy of the system and the energy of the system can be zero or greater than zero but it cannot be negative. So, for stable system the Lyapunov function should be positive definite and its derivative should be negative definite which denotes that energy of the system is continuously decreasing along any trajectory with respect of time.

Sr. No	Indirect Method	Direct Method
	Linear System	**Linear System**
1.	$\ddot{x} + \dot{x} + x = 0$ System proved stable	$\ddot{e} + (k + \alpha^2)e = 0$ System proved stable

2.	$10x_1^2 + 4x_2^2 + x_3^2 + 2x_1x_2 - 2x_2x_3 - 4x_3x_1$ System proved stable	
3.	$\dot{x}_1 = x_2$ $\dot{x}_2 = -2x_2 + x_3$ $\dot{x}_3 = -Kx_1 - x_3 + Ku(t)$ System is stable for the range 0<K<6	
	Nonlinear System	**Nonlinear System**
4.	$\dot{x}_1 = x_2$ $\dot{x}_2 = -x_2 - Kx_1^3$ System is stable for the range $-\frac{1}{\sqrt{3k}} < x_1 < \frac{1}{\sqrt{3k}}$	$\dot{x}_1 = x_2$ $\dot{x}_2 = -2x_1 - x_2^3$ System proved stable
5.	$\dot{x}_1 = x_2$ $\dot{x}_2 = -6x_1 - 5x_2$ $\dot{x}_3 = -2x_2 + x_3$ System proved indefinite then it is not possible to decide about stability.	$\dot{x}_1 = x_2$ $\dot{x}_2 = -6x_1 - 5x_2$ $\dot{x}_3 = -2x_2 + x_3$ System proved indefinite.

Table1. Analysis of Indirect and Direct Methods

6. Comparison, advantages and disadvantages

After implementation of indirect and direct methods on different system, identified advantages and disadvantages related to theorems are as below:

Direct Method	Indirect Method
This method can be applied directly to a nonlinear system without the need to linearization.	This method cannot be applied directly to a nonlinear system.
Energy dissipation with respect to time is the main concept behind this method.	System linearization around a given point is the main idea behind this method.
Global stability can be achieved with this method.	Local stability with small stability regions can be achieved with this method.
Provides radius of convergence.	Indirect Method does not provide a radius of convergence
Need to find a Lyapunov function for this method which is not easy.	No need to find lyapunov function for this method.

Table2.Comparison between Indirect and Direct Methods

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